SPECTRAL LARGE SIEVE INEQUALITIES FOR HECKE CONGRUENCE SUBGROUPS OF $SL(2,\mathbb{Z}[i])$

by

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Abstract: We prove, in respect of an arbitrary Hecke congruence subgroup $\Gamma = \Gamma_0(q_0) \leq SL(2,\mathbb{Z}[i])$, some new upper bounds (or 'spectral large sieve inequalities') for sums involving Fourier coefficients of Γ-automorphic cusp forms on $SL(2,\mathbb{C})$. The Fourier coefficients in question may arise from the Fourier expansion at any given cusp $\mathfrak{c}$ of Γ (our results are not limited to the case $\mathfrak{c} = \infty$). For this reason, our proof is reliant upon an extension, to arbitrary cusps, of the spectral-Kloosterman sum formula for $\Gamma\backslash SL(2,\mathbb{C})$ obtained by Hristina Lokvenec-Guleska in her doctoral thesis (generalising the sum formulae of Roelof Bruggeman and Yoichi Motohashi for $PSL(2,\mathbb{Z}[i])\backslash PSL(2,\mathbb{C})$ in several respects, though not as regards the choice of cusps). A proof of the required extension of the sum formula is given in an appendix.

Keywords: spectral theory, large sieve, Hecke congruence group, Gaussian integers, sum formula, automorphic form, cusp form, non-holomorphic modular form, Fourier coefficient, Kloosterman sum.

Contents

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Outline of results and methods.

In 1982, in Theorem 2 of their paper [9], Deshouillers and Iwaniec generalised Iwaniec's ground-breaking estimate [20], Theorem 1, to obtain similar 'spectral large sieve inequalities' for Fourier coefficients of holomorphic cusp forms, non-holomorphic cusp forms, or Eisenstein series, automorphic (or 'modular') with respect to the action of an abitrary Hecke congruence subgroup of $SL(2,\mathbb{Z})$ upon the upper half complex plane $\mathbb{H}_2$. This paper concerns corresponding results for Fourier coefficients of functions on $SL(2,\mathbb{C})$ that are automorphic with respect to some Hecke congruence subgroup Γ of $SL(2,\mathbb{Z}[i])$. Our principal results, obtained in Theorem 1 below, are not quite a perfect analogy of the results (1.28) and (1.29) of Theorem 2 of [9], and so seem open to further improvement. Another question left open is as to whether it is possible to achieve a refinement of our Theorem 1 paralleling the significant refinement of Deshouillers and Iwaniec's spectral large sieve inequalities that was obtained by Jutila in Theorem 1.1 of [23].

We have modelled our proof of Theorem 1 on the proof of Theorem 2 of [9] that is contained in Section 5 of [9]. Just as the proof of Theorem 2 of [9] is dependent on the estimates for sums of generalised Kloosterman that are supplied in Proposition 3 of [9], so too is our proof of Theorem 1 dependent on the estimates for sums of Kloosterman sums that we obtain in our Proposition 2 (stated at the end of Section 1). Our proof of Proposition 2 follows the same basic pattern as the proof (in [9], Subsection 5.1) of Proposition 3 of [9], but it does have some novel features (such as those relating to the 'grossencharakter' factor $(\omega_1\omega_2/|\omega_1\omega_2|)^m$ which occurs in Equation (1.9.25), below).

In the work [9] of Deshouillers and Iwaniec a crucial part is played by summation formulae of Bruggeman [3] and Kuznetsov [28], [29], expressing certain sums involving Fourier coefficients of modular forms in terms of sums of Kloosterman sums (and vice versa). These particular summation formulae apply only to modular forms on $\mathbb{H}_2$ (i.e. the homogeneous space $SL(2,\mathbb{R})/SO(2,\mathbb{R})$), whereas in the present work one needs instead summation formulae for Fourier coefficients of automorphic functions on $SL(2,\mathbb{C})$. Such formulae were first obtained (for functions automorphic with respect to $SL(2,\mathbb{Z}[i])$) in Bruggeman and Motohashi's paper [5], and were extended to the case of functions automorphic with respect to Hecke congruence subgroups of $SL(2,\mathbb{Z}[i])$ in Lokvenec-Guleska's thesis [32]. These authors dealt only with Fourier coefficients for the cusp at infinity, but their methods can be adapted to succesfully handle Fourier coefficients at other cusps: the relevant summation formula, Theorem B, is stated at the beginning of Subsection 1.9 of this paper; the required adaptations of the proofs in [5] and [32] are discussed in an appendix (Section 6).

An introduction to relevant concepts and terminology now follows (preparatory to the statement of the principal new results contained in this paper).

§1. Definitions and statements of the results.

§1.1 The quotient $\Gamma\backslash G$: coordinates, measure, cusps and fundamental domains.

Let $G = SL(2,\mathbb{C})$ and $K = SU(2)$ (the maximal compact subgroup of the Lie group G). Let $\mathfrak{O}$ denote the ring $\mathbb{Z}[i]$ of Gaussian integers; we shall use the notation $m \sim n$ to signify that m and n are associates (in the sense that $m \in n\mathfrak{O}^* \subset \mathfrak{O}$, where $\mathfrak{O}^* = \{i,-1,-i,1\}$). Suppose moreover that $q_0 \in \mathfrak{O} - \{0\}$; and that Γ is the Hecke congruence subgroup of $SL(2,\mathfrak{O})$ given by

$$\Gamma = \Gamma_0(q_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathfrak{O}) : c \in q_0\mathfrak{O} \right\} . \tag{1.1.1}$$

The group Γ is a discrete and cofinite (but not cocompact) subgroup of G. Both Γ and G act by left multiplication on the homogeneous space G/K. By the Iwasawa decomposition each $g \in G$ has a unique factorisation of form

$$g = n[z]a[r]k[\alpha,\beta] , \tag{1.1.2}$$

where, for $z \in \mathbb{C}$, $r > 0$, and $\alpha,\beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, one has:

$$n[z] = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} , \quad a[r] = \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix} \quad \text{and} \quad k[\alpha,\beta] = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in K . \tag{1.1.3}$$

The mapping $gK \mapsto (z,r)$ (with z and r are as in (1.1.2)) defines a homeomorphism between G/K and the topological space $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}^+ = \mathbb{C} \times \{r \in \mathbb{R} : r > 0\}$. The action of G upon G/K (by left multiplication)

may be interpreted as a continuous group action of G upon $\mathbb{H}_3$ by putting $g(z,r)=(z',r')$ when $z,z'\in\mathbb{C}$, $r,r'>0$ and $gn[z]a[r]K=n[z']a[r']K$: by a calculation, one then has

$$g(z,r)=\left(\frac{(az+b)\,\overline{(cz+d)}+a\,\overline{c}\,r^2}{|cz+d|^2+|c|^2r^2}\,,\,\frac{r}{|cz+d|^2+|c|^2r^2}\right)\in\mathbb{H}_3\quad\text{for }g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G,\ (z,r)\in\mathbb{H}_3. \tag{1.1.4}$$

By Proposition 7.3.9 of [11], the set

$$\mathcal{F}_{\mathbb{Q}(i)}=\left\{(z,r)\in\mathbb{H}_3: |z|^2+r^2\geq 1 \text{ and } |\mathrm{Re}(z)|,\mathrm{Im}(z)\in[0,1/2]\right\} \tag{1.1.5}$$

is a fundamental domain for the action of $SL(2,\mathfrak{O})=\Gamma_0(1)$ upon $\mathbb{H}_3$. Consideration of the natural homomorphism from $SL(2,\mathfrak{O})$ into $SL(2,\mathfrak{O}/q_0\mathfrak{O})$ shows that

$$[SL(2,\mathfrak{O}):\Gamma]=|q_0|^2\prod_{(\varpi)\ni q_0}\left(1+\frac{1}{|\varpi|^2}\right)\,, \tag{1.1.6}$$

where the product is taken over prime ideals $(\varpi)=\varpi\mathfrak{O}\subset\mathfrak{O}$ (see Section 2.4 of [21] for the $SL(2,\mathbb{Z})$-analogue of this). Therefore there exist $\gamma_1,\ldots,\gamma_{[SL(2,\mathfrak{O}):\Gamma]}\in SL(2,\mathfrak{O})$ such that the set

$$\mathcal{F}=\bigcup_{k=1}^{[SL(2,\mathfrak{O}):\Gamma]}\gamma_k\mathcal{F}_{\mathbb{Q}(i)} \tag{1.1.7}$$

is a fundamental domain for the action of Γ upon $\mathbb{H}_3$. Moreover, as noted in Section 2.2 of [11], there do exist fundamental domains for the action of Γ upon $\mathbb{H}_3$ that have a connected interior (these being the Poincaré normal polyhedrons $\mathcal{P}_Q(\Gamma)\subset\mathbb{H}_3$ centred at $Q=(z,r)$). Therefore one may assume a choice of $\gamma_1,\ldots,\gamma_{[SL(2,\mathfrak{O}):\Gamma]}$ in (1.1.7) that makes the interior of $\mathcal{F}$ be connected.

When $k=k[\alpha,\beta]\in K$ one has, for some $\theta\in[0,\pi)$ and some real φ and ψ satisfying $\varphi\pm\psi\in[0,4\pi)$, the factorisation

$$k=h\left[e^{i\varphi/2}\right]v[i\theta]h\left[e^{i\psi/2}\right]\,, \tag{1.1.8}$$

where

$$h[u]=\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix}\qquad\text{and}\qquad v[i\theta]=\begin{pmatrix}\cos(\theta/2)&i\sin(\theta/2)\\i\sin(\theta/2)&\cos(\theta/2)\end{pmatrix}\,. \tag{1.1.9}$$

This factorisation is unique in the cases where θ satisfies $0<\theta<\pi$, so the 'Iwasawa coordinates' z,r,θ,φ,ψ in (1.1.2) and (1.1.8) are a serviceable coordinate system for G.

We define $C^\infty(G)$ to be the space of functions $f:G\to\mathbb{C}$ which are 'smooth', in the sense that, for each $g_0\in G$, and each $j\in\mathbb{N}$, all 2^j of the partial derivatives of order j of the function $(x,y,r,\theta,\varphi,\psi)\mapsto f(g_0n[x+iy]a[r]h[e^{i\varphi/2}]v[i\theta]h[e^{i\psi/2}])$ are defined and continuous on $\mathbb{R}^5$. The space of smooth complex-valued functions on K is denoted by $C^\infty(K)$ (one has $F\in C^\infty(K)$ if and only if F is the restriction to K of some element of $C^\infty(G)$). We define $C^0(G)$ to be the space of functions $f:G\to\mathbb{C}$ that are continuous with respect to the topology on G defined in Section 2.1 of [11] (this just means that a function $f:G\to\mathbb{C}$ will lie in $C^0(G)$ if and only if $f(g)$ is continuous as a function of the Iwasawa coordinates of g).

In terms of the parameterisations introduced in (1.1.3) and (1.1.8)-(1.1.9), the groups

$$K=\left\{k[\alpha,\beta]:\alpha,\beta\in\mathbb{C}\text{ and }|\alpha|^2+|\beta|^2=1\right\},\quad A=\{a[r]:r>0\}\quad\text{and}\quad N=\{n[z]:z\in\mathbb{C}\}$$

have left and right Haar measures

$$\mathrm{d}k=2^{-3}\pi^{-2}\sin(\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\psi,\quad \mathrm{d}a=r^{-1}\,\mathrm{d}r\quad\text{and}\quad \mathrm{d}n=\mathrm{d}_+z=\mathrm{d}x\,\mathrm{d}y\,, \tag{1.1.10}$$

respectively, where x and y are the real and imaginary parts of z. Note that the compactness of K has here allowed a choice of $\mathrm{d}k$ such that

$$\int_K\mathrm{d}k=2\,.$$

The group G also has a left and right Haar measure:

$$\mathrm{d}g = r^{-2}\,\mathrm{d}n\,\mathrm{d}a\,\mathrm{d}k = r^{-3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}k \quad \text{for } g = n[x+iy]a[r]k \text{ with } x,y\in\mathbb{R},\ r>0 \text{ and } k\in K\ . \tag{1.1.11}$$

With respect to the hyperbolic Riemannian metric on $\mathbb{H}_3$,

$$\frac{|\mathrm{d}z|^2+\mathrm{d}r^2}{r^2} = \frac{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}r^2}{r^2}\ , \tag{1.1.12}$$

the elements of G act upon $\mathbb{H}_3$ as elements of the group $Iso^+(\mathbb{H}_3)$ of orientation preserving isometries: one has in effect a homomorphism $g\mapsto g|_{\mathbb{H}_3}$ from G into $Iso^+(\mathbb{H}_3)$, which (see [11], Proposition 1.1.3) is surjective and has kernel $\{h[1],h[-1]\}$. The hyperbolic metric (1.1.12) makes $\mathbb{H}_3$ a model for three dimensional hyperbolic space, and induces on $\mathbb{H}_3$ a G-invariant measure,

$$r^{-3}\,\mathrm{d}_+z\,\mathrm{d}r = r^{-3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r = \mathrm{d}Q \qquad (Q=(z,r)=(x+iy,r) \text{ with } x,y\in\mathbb{R} \text{ and } r>0), \tag{1.1.13}$$

identical to that which is induced (via the homeomorphism between G/K and $\mathbb{H}_3$) by the Haar measure (1.1.11). Let $K^+\subset K$ be a fundamental domain for $\{h[1],h[-1]\}\backslash K$; and $\mathcal{F}'$ any fundamental domain for the action of Γ on $\mathbb{H}_3$. Then a fundamental domain for $\Gamma\backslash G$ is the set $\{n[z]a[r]:(z,r)\in\mathcal{F}'\}K^+\subset NAK=G$. One therefore has (using the fundamental domain $\mathcal{F}$ from (1.1.7)):

$$\mathrm{vol}(\Gamma\backslash G) = \int_{\Gamma\backslash G}\mathrm{d}g = \int_{\mathcal{F}}\mathrm{d}Q\int_{K^+}\mathrm{d}k = \int_{\mathcal{F}}\mathrm{d}Q = \mathrm{vol}(\mathcal{F})\ , \tag{1.1.14}$$

where, by (1.1.5)-(1.1.7) and Theorem 7.1.1 of [11],

$$\mathrm{vol}(\mathcal{F}) = \int_{\mathcal{F}} r^{-3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r = \mathrm{vol}\left(\mathcal{F}_{\mathbb{Q}(i)}\right)[SL(2,\mathfrak{O}):\Gamma] = 2\pi^{-2}\zeta_{\mathbb{Q}(i)}(2)\,|q_0|^2\prod_{\substack{\mathfrak{O}\supset(\varpi)\ni q_0\\ (\varpi)\text{ is prime}}}\left(1+\frac{1}{|\varpi|^2}\right) \tag{1.1.15}$$

with $\zeta_{\mathbb{Q}(i)}(s)=\zeta(s)L(s,\chi_4)$ being the Dedekind zeta-function for $\mathbb{Q}(i)$, so that

$$2\pi^{-2}\zeta_{\mathbb{Q}(i)}(2) = \frac{2}{\pi^2}\sum_{0\neq\alpha\in\mathfrak{O}}|\alpha|^{-4} = \frac{1}{3}L(2,\chi_4) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} = \frac{1}{3}\left(1-\sum_{j=1}^{\infty}2^{-4j}j\zeta(2j+1)\right)$$

(see Page 312 of [11]).

The actions of elements of G upon $\mathbb{H}_3$ extend, by continuity, to actions upon $\mathbb{H}_3\cup\mathbb{P}^1(\mathbb{C})$, where $\mathbb{P}^1(\mathbb{C})=\mathbb{C}\cup\{\infty\}$ (the Riemann sphere): a projective point $\mathfrak{z}=[z_1,z_2]\in\mathbb{P}^1(\mathbb{C})$ being mapped by the action of

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G$$

to the point $g\mathfrak{z}=[az_1+bz_2,cz_1+dz_2]\in\mathbb{P}^1(\mathbb{C})$ (so that $g\infty=\infty$ if and only if $c=0$). If $Q=(z,r)\in\mathbb{H}_3$, then the subgroup of elements of Γ fixing Q is finite. In contrast there exist points $\mathfrak{z}\in\mathbb{P}^1(\mathbb{C})$ for which the stabiliser, $\Gamma_{\mathfrak{z}}=\{g\in\Gamma: g\mathfrak{z}=\mathfrak{z}\}$, contains a free Abelian subgroup of rank 2: such points $\mathfrak{z}$ are called 'cusps' of Γ. Since Γ is a congruence subgroup (that is, there exists $M\in\mathfrak{O}-\{0\}$, namely $M=q_0$, such that Γ contains the kernel of the natural homomorphism from $SL(2,\mathfrak{O})$ into $SL(2,\mathfrak{O}/M\mathfrak{O})$), the set of all cusps of Γ is simply $\mathbb{Q}(i)\cup\{\infty\}=\mathbb{P}^1(\mathbb{Q}(i))$.

Each point $Q\in\mathbb{H}_3$, or $\mathfrak{z}\in\mathbb{P}^1(\mathbb{C})$, has a Γ-orbit: $\Gamma Q=\{\gamma Q:\gamma\in\Gamma\}\subset\mathbb{H}_3$, $\Gamma\mathfrak{z}=\{\gamma\mathfrak{z}:\gamma\in\Gamma\}\subset\mathbb{P}^1(\mathbb{C})$. For $Q\in\mathbb{H}_3$ one has $1\leq|\mathcal{F}\cap\Gamma Q|\ll 1$ and $|\mathrm{Int}(\mathcal{F})\cap\Gamma Q|\leq 1$. For a pair of cusps $\mathfrak{a},\mathfrak{b}\in\mathbb{P}^1(\mathbb{Q}(i))$, the relation $\mathfrak{a}\overset{\Gamma}{\sim}\mathfrak{b}$ ('Γ-equivalence') is deemed to hold if and only if $\Gamma\mathfrak{a}=\Gamma\mathfrak{b}$. This is an equivalence relation under which $\mathbb{P}^1(\mathbb{Q}(i))$ is partitioned into a finite number of distinct equivalence classes, $\mathcal{P}_1,\ldots,\mathcal{P}_{H(\Gamma)}$.

Let $\mathfrak{c}$ be a cusp for Γ. Since G acts transitively (even 3-transitively) on $\mathbb{P}^1(\mathbb{C})$, one may choose $g_{\mathfrak{c}} \in G$ such that

$$g_{\mathfrak{c}}\infty = \mathfrak{c} \; . \tag{1.1.16}$$

One then has

$$\Gamma_{\mathfrak{c}} = \Gamma \cap g_{\mathfrak{c}} P g_{\mathfrak{c}}^{-1} \; , \tag{1.1.17}$$

where

$$P = \{g \in G : g\infty = \infty\} = \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} : u \in \mathbb{C}^*, \; v \in \mathbb{C} \right\} \; . \tag{1.1.18}$$

The maximal free abelian subgroup $\Gamma'_{\mathfrak{c}} \leq \Gamma_{\mathfrak{c}}$ consists of the identity element and unipotent elements (non-identity elements with trace equal to 2). By (1.1.17),

$$\Gamma'_{\mathfrak{c}} = \Gamma \cap g_{\mathfrak{c}} N g_{\mathfrak{c}}^{-1} \; . \tag{1.1.19}$$

Note that $\Gamma'_{\mathfrak{c}}$ is a normal subgroup of $\Gamma_{\mathfrak{c}}$: for if $\gamma \in \Gamma'_{\mathfrak{c}}$ and $\eta \in \Gamma_{\mathfrak{c}}$ then $\mathrm{Tr}\big(\eta\gamma\eta^{-1}\big) = \mathrm{Tr}(\gamma) = 2$.

As is shown in Lemma 4.2 of this paper, the above 'scaling matrix' $g_{\mathfrak{c}} \in G$ may be chosen so that one has both (1.1.16) and

$$g_{\mathfrak{c}}^{-1}\Gamma'_{\mathfrak{c}} g_{\mathfrak{c}} = g_{\mathfrak{c}}^{-1}\Gamma g_{\mathfrak{c}} \cap N = B^+ \; , \tag{1.1.20}$$

where

$$B^+ = \{n[\alpha] : \alpha \in \mathfrak{O}\} = SL(2,\mathfrak{O})'_{\infty} \; . \tag{1.1.21}$$

Such choice of $g_{\mathfrak{c}}$ simplifies Fourier expansions at cusps: see (1.4.1)-(1.4.3) below. It is therefore to be assumed throughout this paper that one works with a choice of scaling matrices such that (1.1.16)-(1.1.21) hold for all cusps $\mathfrak{c}$ of Γ.

For all cusps $\mathfrak{c}$ of Γ one has $[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}] \in \{2,4\}$ and, by appropriate choice of scaling matrix $g_{\mathfrak{c}}$ (satisfying (1.1.16) and (1.1.20)-(1.1.21)), one may ensure that $g_{\mathfrak{c}}^{-1}\Gamma_{\mathfrak{c}} g_{\mathfrak{c}}|_{\mathbb{P}^1(\mathbb{C})-\{\infty\}}$ has as a fundamental domain the set

$$\mathcal{R}_{\mathfrak{c}} = \begin{cases} \{z \in \mathbb{C} : |\mathrm{Re}(z)| \leq 1/2, \; |\mathrm{Im}(z)| \leq 1/2\} & \text{if } [\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}] = 2 \; , \\ \{z \in \mathbb{C} : |\mathrm{Re}(z)| \leq 1/2, \; 0 \leq \mathrm{Im}(z) \leq 1/2\} & \text{if } [\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}] = 4 \; . \end{cases} \tag{1.1.22}$$

Any cusp $\mathfrak{c} = u/w \in \mathbb{Q}(i)$ (with $u, w \in \mathfrak{O}$, $w \neq 0$ and $(u,w) \sim 1$) has a 'width' $|m_{\mathfrak{c}}|^2$, where $m_{\mathfrak{c}} \sim q_0/\big(w^2, q_0\big)$. By defining $m_\infty \sim 1$ one ensures that each pair of Γ-equivalent cusps $\mathfrak{a}, \mathfrak{b}$ has $m_{\mathfrak{a}} \sim m_{\mathfrak{b}}$, and equal widths.

For a suitable set of representatives $\mathfrak{C}(\Gamma) \subset \mathbb{Q}(i) \cup \{\infty\}$ of the Γ-equivalence classes of cusps, and suitably chosen scaling matrices $g_{\mathfrak{c}}$ ($\mathfrak{c} \in \mathfrak{C}(\Gamma)$) satisfying (1.1.16) and (1.1.20)-(1.1.21), the sets

$$\mathcal{E}_{\mathfrak{c}} = \{g_{\mathfrak{c}} Q : Q = (z,r) \in \mathbb{H}_3, z \in \mathcal{R}_{\mathfrak{c}} \text{ and } r > 1/\,|m_{\mathfrak{c}}|\} \qquad (\mathfrak{c} \in \mathfrak{C}(\Gamma)) \tag{1.1.23}$$

are pairwise disjoint non-compact subsets of $\mathbb{H}_3$ and, for some compact hyperbolic polyhedron $\mathcal{D} \subset \mathbb{H}_3$ (having finitely many faces and a connected interior), the union of sets

$$\mathcal{F}_* = \mathcal{D} \cup \bigcup_{\mathfrak{c} \in \mathfrak{C}(\Gamma)} \mathcal{E}_{\mathfrak{c}} \tag{1.1.24}$$

is a fundamental domain for the action of Γ on $\mathbb{H}_3$, with a connected interior, $\mathrm{Int}(\mathcal{F}_*)$ (it simultaneously being the case that, for $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, one has $\mathrm{Int}(\mathcal{D}) \cap \big(\overline{\mathcal{E}_{\mathfrak{c}}} - \mathrm{Int}\big(\mathcal{E}_{\mathfrak{c}}\big)\big) = \mathcal{E}'_{\mathfrak{c}} - \mathcal{E}^*_{\mathfrak{c}}$, where the set $\mathcal{E}^*_{\mathfrak{c}}$ is finite and $\mathcal{E}'_{\mathfrak{c}} = \big\{g_{\mathfrak{c}} Q : Q = (z, 1/|m_{\mathfrak{c}}|) \in \mathbb{H}_3 \, , \, z \in \mathrm{Int}\big(\mathcal{R}_{\mathfrak{c}}\big)\big\}$). We make use of this type of fundamental domain in the appendix to this paper: see the proof of Corollary 6.2.10, and the proof of Lemma 6.5.16.

§1.2 Γ-automorphic functions, Casimir operators and the Laplacian for $\mathbb{H}_3$.

A function $f : G \to \mathbb{C}$ is said to be Γ-automorphic if and only if it satisfies

$$f(\gamma g) = f(g) \quad \text{for } \gamma \in \Gamma \text{ and } g \in G\,. \tag{1.2.1}$$

Since $\Gamma \ni h[-1]$, the Γ-automorphic functions f are even (i.e. for $g \in G$ they satisfy $f(h[-1]g) = f(g)$). The 'square integrable' Γ-automorphic functions $f : G \to \mathbb{C}$ are those satisfying $\langle f, f\rangle_{\Gamma\backslash G} < \infty$, where

$$\langle f, h\rangle_{\Gamma\backslash G} = \int_{\Gamma\backslash G} f(g)\overline{h(g)}\mathrm{d}g = \int_{\mathcal{F}}\int_{K^+} f(n[z]a[r]k)\overline{h(n[z]a[r]k)}\,\mathrm{d}k\, r^{-3}\mathrm{d}_+z\,\mathrm{d}r\ . \tag{1.2.2}$$

The space $L^2(\Gamma\backslash G)$ of all square integrable Γ-automorphic functions is (if one does not discriminate between functions that are equal almost everywhere) a Hilbert space with respect to the inner product in (1.2.2). For $f \in L^2(\Gamma\backslash G)$, the norm $\|f\|_{\Gamma\backslash G}$ of f is given by $\|f\|_{\Gamma\backslash G} = \sqrt{\langle f, f\rangle_{\Gamma\backslash G}}$. The space of all smooth Γ-automorphic functions on G is

$$C^\infty(\Gamma\backslash G) = \{f \in C^\infty(G) : f \text{ is } \Gamma\text{-automorphic}\}\ . \tag{1.2.3}$$

By the above definitions, neither of the spaces $L^2(\Gamma\backslash G)$ or $C^\infty(\Gamma\backslash G)$ contains the other, and all functions contained in $L^2(\Gamma\backslash G) \cup C^\infty(\Gamma\backslash G)$ are measurable (i.e. measurable with respect to the Haar measure $\mathrm{d}g$).

The Γ-automorphic functions on $\mathbb{H}_3$ are those complex-valued functions that satisfy

$$f(\gamma Q) = f(Q) \quad \text{for } \gamma \in \Gamma \text{ and } Q \in \mathbb{H}_3\,; \tag{1.2.4}$$

and of these, those that have $\int_{\mathcal{F}} |f(Q)|^2\,\mathrm{d}Q < \infty$ are the elements of the Hilbert space $L^2\left(\Gamma\backslash\mathbb{H}_3\right)$. The norm on $L^2\left(\Gamma\backslash\mathbb{H}_3\right)$ is given by $\|f\|_{\Gamma\backslash\mathbb{H}_3} = \sqrt{\langle f, f\rangle_{\Gamma\backslash\mathbb{H}_3}}$, where

$$\langle f_1, f_2\rangle_{\Gamma\backslash\mathbb{H}_3} = \int_{\mathcal{F}} f_1(Q)\overline{f_2(Q)}\,\mathrm{d}Q \quad \text{for } f_1, f_2 \in L^2\left(\Gamma\backslash\mathbb{H}_3\right). \tag{1.2.5}$$

As explained just prior to (1.2.13), below, the space $C^\infty\left(\mathbb{H}_3\right)$ of infinitely differentiable functions $f : \mathbb{H}_3 \mapsto \mathbb{C}$ may be viewed as a certain subpace of 'K-trivial' functions contained in $C^\infty(G)$. One may similarly view $L^2\left(\Gamma\backslash\mathbb{H}_3\right)$ as the subspace of K-trivial functions in $L^2\left(\Gamma\backslash G\right)$.

The complex Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{sl}(2,\mathbb{C})$ is the real vector space of all complex 2×2 matrices with trace equal to zero. The elements of $\mathfrak{g}$ may be identified with left-invariant first order differential operators on $C^\infty(G)$ by setting

$$(\mathbf{X}f)(g) = \frac{\mathrm{d}}{\mathrm{d}t} f\left(g\exp(t\mathbf{X})\right)\big|_{t=0} \quad \text{for } \mathbf{X} \in \mathfrak{sl}(2,\mathbb{C}),\ f \in C^\infty(G) \text{ and } g \in G \tag{1.2.6}$$

(where $\mathrm{d}/\mathrm{d}t$ signifies differentiation of a function of a real variable). Then the universal enveloping algebra, $\mathcal{U}(\mathfrak{g}) \supset \mathfrak{g}$, has centre $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}\left[\Omega_+, \Omega_-\right]$, where, in terms of the Iwasawa coordinates, one has

$$\begin{aligned}
\Omega_+ &= F^+_{r,\varphi,\theta}\!\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}\right) = \\
&= F^+_{r,\varphi,\theta}\!\left(\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}\right) = \\
&= \frac{1}{2}r^2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} + \frac{1}{2}re^{i\varphi}\cot(\theta)\frac{\partial}{\partial z}\frac{\partial}{\partial \varphi} - \frac{1}{2}ire^{i\varphi}\frac{\partial}{\partial z}\frac{\partial}{\partial \theta} - \frac{1}{2}re^{i\varphi}\csc(\theta)\frac{\partial}{\partial z}\frac{\partial}{\partial \psi} + \\
&\qquad + \frac{1}{8}r^2\frac{\partial^2}{\partial r^2} - \frac{1}{4}ir\frac{\partial}{\partial r}\frac{\partial}{\partial \varphi} - \frac{1}{8}\frac{\partial^2}{\partial \varphi^2} - \frac{1}{8}r\frac{\partial}{\partial r} + \frac{1}{4}i\frac{\partial}{\partial \varphi}
\end{aligned} \tag{1.2.7}$$

and $\Omega_- = F^-_{r,\varphi,\theta}\left(\partial/\partial\overline{z}, \partial/\partial z, \partial/\partial r, \partial/\partial\varphi, \partial/\partial\theta, \partial/\partial\psi\right)$ with, for $\varphi, \theta \in \mathbb{R}$ and $r > 0$, each coefficient of $F^-_{r,\varphi,\theta} \in \mathbb{C}\left[X_1, \ldots, X_6\right]$ being equal to the complex-conjugate of the corresponding coefficient in the polynomial $F^+_{r,\varphi,\theta}\left(X_1, \ldots, X_6\right)$. A function $f \in C^\infty(G)$ is said to be a function with character Υ (for $\mathcal{Z}(\mathfrak{g})$) if and only if

$$\Omega_+ f = \Upsilon\left(\Omega_+\right) f \quad \text{and} \quad \Omega_- f = \Upsilon\left(\Omega_-\right) f\ . \tag{1.2.8}$$

The complex Lie algebra of K is $\mathfrak{k} = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{su}(2)$ is the set of skew Hermitian elements of $\mathfrak{sl}(2,\mathbb{C})$. Both it and its universal enveloping algebra, $\mathcal{U}(\mathfrak{k}) \supset \mathfrak{k}$, are generated by the elements

$$\mathbf{H}_2 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad \mathbf{W}_1 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \mathbf{W}_2 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix} . \tag{1.2.9}$$

The elements of $\mathcal{U}(\mathfrak{k})$ may be interpreted (similarly to those of $\mathcal{U}(\mathfrak{g})$) as left-invariant differential operators:

$$(\mathbf{H}_2 f)(k) = \frac{\mathrm{d}}{\mathrm{d}t} f\left(k \exp(t\mathbf{H}_2)\right)\big|_{t=0} = \left(\frac{\partial f}{\partial \psi}\right)(k) \quad \text{for } f \in C^\infty(K) \text{ and } k \in K \tag{1.2.10}$$

(for example). The centre of $\mathcal{U}(\mathfrak{k})$ is $\mathcal{Z}(\mathfrak{k}) = \mathbb{C}\left[\Omega_{\mathfrak{k}}\right]$, where

$$\Omega_{\mathfrak{k}} = \frac{1}{2}\left(\mathbf{H}_2^2 + \mathbf{W}_1^2 + \mathbf{W}_2^2\right) = \frac{1}{2}\csc^2(\theta)\left(\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\psi^2}\right) - \csc(\theta)\cot(\theta)\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\psi} + \frac{1}{2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{2}\cot(\theta)\frac{\partial}{\partial\theta} . \tag{1.2.11}$$

In addition to being left-invariant the Casimir operators $\Omega_\pm$ are also right-invariant:

$$R_g\Omega_+ f = \Omega_+ R_g f, \quad \text{and} \quad R_g\Omega_- f = \Omega_- R_g f \quad \text{for } g \in G \text{ and } f \in C^\infty(G)\,, \tag{1.2.12}$$

where R_g (the right-translation operator) maps f to the function $R_g f \in C^\infty(G)$ such that $(R_g f)(h) = f(hg)$ for $h \in G$. This can be proved by the methods of [6], Proposition 2.2.4 and Lemma 2.2.2, where an analogous result concerning the centre of $\mathcal{U}(\mathfrak{gl}(n,\mathbb{R}))$ is obtained. One can show likewise that $\Omega_{\mathfrak{k}}$ is an operator on $C^\infty(K)$ that is invariant with respect to right-translation by any element of K.

By (1.2.12) whenever $f \in C^\infty(G/K) = \{\phi \in C^\infty(G) : \phi(gk) = \phi(g) \text{ for all } k \in K, g \in G\}$ (the space of 'K-trivial' elements of $C^\infty(G)$) one will then also have $\Omega_\pm f \in C^\infty(G/K)$. Therefore, and by virtue of the natural bijection (induced by the homeomorphism between G/K and $\mathbb{H}_3$ described below (1.1.3)) from $C^\infty(G/K)$ onto $C^\infty(\mathbb{H}_3)$, one may view $\Omega_+|_{C^\infty(G/K)}$ and $\Omega_-|_{C^\infty(G/K)}$ as being operators from $C^\infty(\mathbb{H}_3)$ into $C^\infty(\mathbb{H}_3)$. The hyperbolic Laplacian operator on $C^\infty(G/K)$, or (equivalently) on $C^\infty(\mathbb{H}_3)$, is then

$$\mathbf{\Delta} = 4\left(\Omega_+ + \Omega_-\right)\big|_{C^\infty(G/K)} = r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) - r\frac{\partial}{\partial r}\,, \tag{1.2.13}$$

where x, y, r signify real-valued coordinates of a point $(x+iy, r) \in \mathbb{H}_3$. This Laplacian inherits, from the Casimir operators, left-invariance with respect to the actions of elements of G, so that

$$(\mathbf{\Delta} f) \circ (g|_{\mathbb{H}_3}) = \mathbf{\Delta}\left(f \circ (g|_{\mathbb{H}_3})\right) \quad \text{for } f \in C^\infty(\mathbb{H}_3) \text{ and } g \in G\,. \tag{1.2.14}$$

Let

$$C^\infty(\Gamma\backslash\mathbb{H}_3) = \{f \in C^\infty(\mathbb{H}_3) : f \text{ is } \Gamma\text{-automorphic}\}\ . \tag{1.2.15}$$

Then by the restriction of (1.2.14) to $g \in \Gamma$, one has

$$\mathbf{\Delta} f \in C^\infty(\Gamma\backslash\mathbb{H}_3) \quad \text{if} \quad f \in C^\infty(\Gamma\backslash\mathbb{H}_3)\,. \tag{1.2.16}$$

Similarly, by the left-invariance of the elements of $\mathfrak{g}$ (viewed as differential operators on $C^\infty(G)$),

$$\Psi f \in C^\infty(\Gamma\backslash G) \quad \text{if} \quad \Psi \in \mathcal{U}(\mathfrak{g}) \quad \text{and} \quad f \in C^\infty(\Gamma\backslash G)\,. \tag{1.2.17}$$

If $f, \phi \in L^2(\Gamma\backslash\mathbb{H}_3) \cap C^\infty(\mathbb{H}_3)$ are such that $\mathbf{\Delta} f, \mathbf{\Delta}\phi \in L^2(\Gamma\backslash\mathbb{H}_3)$, then

$$\langle -\mathbf{\Delta} f, \phi\rangle_{\Gamma\backslash\mathbb{H}_3} = \int_\phi \left(\frac{\partial f}{\partial x}\overline{\frac{\partial \phi}{\partial x}} + \frac{\partial f}{\partial y}\overline{\frac{\partial \phi}{\partial y}} + \frac{\partial f}{\partial r}\overline{\frac{\partial \phi}{\partial r}}\right) r^2 \mathrm{d}Q\,, \tag{1.2.18}$$

so that when operating on functions satisfying the above constraints $-\mathbf{\Delta}$ is both symmetric and positive:

$$\langle -\mathbf{\Delta} f, \phi\rangle_{\Gamma\backslash\mathbb{H}_3} = \langle f, -\mathbf{\Delta}\phi\rangle_{\Gamma\backslash\mathbb{H}_3}\,, \tag{1.2.19}$$

and

$$\langle -\mathbf{\Delta} f, f\rangle_{\Gamma\backslash\mathbb{H}_3} > 0 \quad \text{if } f \text{ is non-constant.} \tag{1.2.20}$$

These results (1.2.18)-(1.2.20) are contained in Theorem 4.1.7 of [11], where the (effectively) more general case of an arbitrary discrete subgoup $\Gamma \leq PSL(2,\mathbb{C})$ is treated, and where it is moreover shown that for an appropriately extended domain of definition the operator $-\mathbf{\Delta}$ is essentially self-adjoint. As for the Casimir operator $\Omega_{\mathfrak{k}} \in \mathcal{Z}(\mathfrak{k})$, one has by [42], Chapter 2, Equation (6.3), the result that

$$\left(\Omega_{\mathfrak{k}} F, \Phi\right)_K = \left(F, \Omega_{\mathfrak{k}}\Phi\right)_K \quad \text{for } F, \Phi \in C^\infty(K)\,, \tag{1.2.21}$$

where

$$\left(F_1, F_2\right)_K = \int_K F_1(k)\overline{F_2(k)}\,\mathrm{d}k\,. \tag{1.2.22}$$

§1.3 Functions of K-type (ℓ, q).

The differential operators $\mathbf{H}_2$ and $\Omega_{\mathfrak{k}}$, as initially defined, have domain $C^\infty(K)$. One extends $\mathbf{H}_2$ to the domain $C^\infty(G)$ by putting $\left(\mathbf{H}_2 f\right)(g) = \left(\mathbf{H}_2 f_g\right)(k[1,0])$ for $g \in G - K$, where each function $f_g : K \to \mathbb{C}$ is given by $f_g(k) = f(gk)$. The corresponding extension of any other elements of $\mathcal{U}(\mathfrak{k})$ to the domain $C^\infty(G)$ (that of $\Omega_{\mathfrak{k}}$ in particular) is defined similarly. In what follows $\mathbf{H}_2$ and $\Omega_{\mathfrak{k}}$ may denote either the extensions just defined, or their restrictions to the domain $C^\infty(K)$ (i.e. the differential operators $\mathbf{H}_2, \Omega_{\mathfrak{k}}$ as initially defined in (1.2.10), (1.2.11)): in each instance the reader should infer the option suited to the relevant operand.

Assuming $\ell \geq -1/2$ and $q \in \mathbb{R}$, an element f of either $C^\infty(K)$ or $C^\infty(G)$ is said to be 'of K-type (ℓ, q)' if and only if

$$\mathbf{H}_2 f = -iqf \quad \text{and} \quad \Omega_{\mathfrak{k}} f = -\frac{1}{2}\left(\ell^2 + \ell\right) f\,. \tag{1.3.1}$$

If it is the case that $q \in \mathbb{Z}$ then all functions of K-Type (ℓ, q) are necessarily even functions (this follows by virtue of (1.2.10), (1.1.2), (1.1.8) and (1.1.9)). To obtain useful examples of such functions, suppose that $\nu \in \mathbb{C}$, $p, q, \ell \in \mathbb{Z}$, $\ell \geq |p|, |q|$ and $k = k[\alpha, \beta] \in K$, and let the function $\varphi_{\ell,q}(\nu, p) : G \to \mathbb{C}$ be defined by

$$\varphi_{\ell,q}(\nu, p)\,(na[r]k) = r^{1+\nu}\Phi^{\ell}_{p,q}(k) \qquad (n \in N,\ r > 0,\ k \in K), \tag{1.3.2}$$

where $\Phi^{\ell}_{p,q}\left(k[\alpha,\beta]\right)$ is the coefficient of $X^{\ell-p}$ in $(\alpha X - \overline{\beta})^{\ell-q}(\beta X + \overline{\alpha})^{\ell+q} \in \mathbb{C}[X]$. Then $\varphi_{\ell,q}(\nu, p)$ lies in the space

$$C^\infty(N\backslash G) = \{f \in C^\infty(G) : f(ng) = f(g) \text{ for } n \in N,\ g \in G\}$$

and is a function of K-type (ℓ, q) with, moreover, character $\Upsilon = \Upsilon_{\nu,p}$, the unique character for $\mathcal{Z}(\mathfrak{g})$ such that

$$\Upsilon_{\nu,p}\left(\Omega_+\right) = \frac{1}{8}\left((\nu - p)^2 - 1\right) \qquad \text{and} \qquad \Upsilon_{\nu,p}\left(\Omega_-\right) = \frac{1}{8}\left((\nu + p)^2 - 1\right)\,. \tag{1.3.3}$$

For $p, q, \ell \in \mathbb{Z}$ and $\ell \geq |p|, |q|$, the above defined function $\Phi^{\ell}_{p,q}(k)$ is an element of $C^\infty(K)$ of K-type (ℓ, q); it is, moreover, an even function of k: for it follows directly from the definition that

$$\Phi^{\ell}_{p,q}\left(h\left[e^{i\varphi/2}\right]kh\left[e^{i\psi/2}\right]\right) = e^{-ip\varphi - iq\psi}\Phi^{\ell}_{p,q}(k) \qquad (\varphi, \psi \in \mathbb{R} \text{ and } k \in K). \tag{1.3.4}$$

The set $\mathcal{L}_{\text{even}}(K) = \left\{\Phi^{\ell}_{p,q} : p, q, \ell \in \mathbb{Z} \text{ and } \ell \geq |p|, |q|\right\}$ is orthogonal with respect to the inner-product defined in (1.2.22); it spans a dense subspace of the Hilbert space, $L^2_{\text{even}}(K)$, of even functions $f : K \to \mathbb{C}$ such that $\int_K |f|^2 \mathrm{d}k < \infty$. By (1.2.21) and the points since noted (excluding (1.3.3)) one may deduce that if $f \neq 0$ is an even function of K-type (ℓ, q) then $\ell, q \in \mathbb{Z}$, $\ell \geq |q|$ and there exist K-trivial functions $h_{f,p'} \in C^\infty(G/K)$ $(p' = -\ell, \ldots, \ell)$ such that

$$f(nak) = \sum_{p'=-\ell}^{\ell} h_{f,p'}(na)\Phi^{\ell}_{p',q}(k) \quad \text{for } n \in N,\ a \in A \text{ and } k \in K\,. \tag{1.3.5}$$

§1.4 Fourier expansions at cusps; spaces of cusp forms.

A function $f : G \to \mathbb{C}$ is said to be of uniform polynomial growth along A if and only if there exist real numbers $b \geq 0$ and $r_0 \geq 1$ such that

$$|f(na[r]k)| \leq r^b \quad \text{for } n \in N,\ k \in K \text{ and } r \geq r_0\ .$$

A Γ-automorpic function $f : G \to \mathbb{C}$ is said to have polynomial growth if and only if it is that case that, for all cusps $\mathfrak{c}$ of Γ, the function $(f|\mathfrak{c}) : G \to \mathbb{C}$ given by $(f|\mathfrak{c})(g) = f(g_{\mathfrak{c}}g)$ is of uniform polynomial growth along A.

If $\mathfrak{c}$ is a cusp of Γ, and if $f : G \to \mathbb{C}$ is Γ-automorphic, then, by (1.1.19)-(1.1.21) and (1.2.1), the above defined function $(f|\mathfrak{c})$ satisfies $(f|\mathfrak{c})(n[\alpha]g) = (f|\mathfrak{c})(g)$ for all $\alpha \in \mathfrak{O}$ and all $g \in G$. Hence, for any $f \in C^\infty(\Gamma\backslash G)$, one has the Fourier expansion at $\mathfrak{c}$:

$$f(g_{\mathfrak{c}}g) = (f|\mathfrak{c})(n[0]g) = \sum_{\omega\in\mathfrak{O}} \left(F_\omega^{\mathfrak{c}} f\right)(g) \qquad (g \in G), \tag{1.4.1}$$

where $(F_\omega^{\mathfrak{c}} f)(g)$, the 'Fourier term of order ω for f at $\mathfrak{c}$', is given by

$$\left(F_\omega^{\mathfrak{c}} f\right)(g) = \int_{B^+\backslash N} \left(\psi_\omega(n)\right)^{-1} f(g_{\mathfrak{c}}ng)\,\mathrm{d}n \tag{1.4.2}$$

with B^+ as in (1.1.20)-(1.1.21), and

$$\psi_\omega(n[z]) = \mathrm{e}\left(\mathrm{Re}(\omega z)\right) \quad \text{for } \omega \in \mathfrak{O},\ z \in \mathbb{C} \tag{1.4.3}$$

(it being henceforth understood that $\mathrm{e}(\beta) = e^{2\pi i\beta}$ for $\beta \in \mathbb{C}$). For $\ell, q \in \mathbb{Z}$, with $\ell \geq |q|$, and any character Υ of $\mathcal{Z}(\mathfrak{g})$, let

$$A_\Gamma(\Upsilon;\ell,q) = \{f \in C^\infty(\Gamma\backslash G) : f \text{ is a of } K\text{-type } (\ell,q) \text{ with character } \Upsilon\}\ . \tag{1.4.4}$$

Each of these spaces $A_\Gamma(\Upsilon;\ell,q)$ has the subspaces

$$A_\Gamma^{\mathrm{pol}}(\Upsilon;\ell,q) = \{f \in A_\Gamma(\Upsilon;\ell,q) : f \text{ has polynomial growth}\} \tag{1.4.5}$$

and

$$A_\Gamma^0(\Upsilon;\ell,q) = \left\{f \in A_\Gamma^{\mathrm{pol}}(\Upsilon;\ell,q) : \left(F_0^{\mathfrak{c}} f\right)(g) = 0 \text{ for all cusps } \mathfrak{c} \text{ of } \Gamma \text{ and all } g \in G\right\}\ . \tag{1.4.6}$$

The latter of these, $A_\Gamma^0(\Upsilon;\ell,q)$, is the space of all cusp forms of K-type (ℓ,q) with character Υ.

By (1.4.2)-(1.4.3), the operator $F_\omega^{\mathfrak{c}}$ maps each f in the space $C^\infty(\Gamma\backslash G)$ to an even function $F_\omega^{\mathfrak{c}} f$ in the space

$$C^\infty(N\backslash G,\omega) = \{h \in C^\infty(G) : h(ng) = \psi_\omega(n)h(g) \text{ for } n \in N, g \in G\} \tag{1.4.7}$$

and commutes with the actions (as differential operators upon those spaces) of the elements of $\mathcal{U}(\mathfrak{g})$. Consequently $F_\omega^{\mathfrak{c}}$ maps functions $f \in A_\Gamma(\Upsilon;\ell,q)$ to functions $F_\omega^{\mathfrak{c}} f$ lying in the complex vector space

$$W_\omega(\Upsilon;\ell,q) = \{h \in C^\infty(N\backslash G,\omega) : h \text{ is of } K\text{-type } (\ell,q) \text{ with character } \Upsilon\}\ . \tag{1.4.8}$$

If $f \in C^\infty(\Gamma\backslash G)$ has polynomial growth, then by (1.4.2)-(1.4.3) the function $F_\omega^{\mathfrak{c}} f \in C^\infty(N\backslash G,\omega)$ is of uniform polynomial growth along A. Therefore if $f \in A_\Gamma^{\mathrm{pol}}(\Upsilon;\ell,q)$, then $F_\omega^{\mathfrak{c}} f$ lies in the complex vector space

$$W_\omega^{\mathrm{pol}}(\Upsilon;\ell,q) = \{h \in W_\omega(\Upsilon;\ell,q) : h \text{ is of uniform polynomial growth along } A\}\ . \tag{1.4.9}$$

Given the restriction to K-types (ℓ,q) with $\ell, q \in \mathbb{Z}$ (and $\ell \geq q$) it follows by (1.1.8), (1.2.10), (1.3.1) and (1.4.8) that all functions in the space $W_\omega(\Upsilon;\ell,q)$ are even. In [5], Lemma 4.1, Lemma 4.2 and Lemma 4.3, the following is proved. If $W_\omega(\Upsilon;\ell,q) \neq \{0\}$, then there exists $(\nu,p) \in \mathbb{C}\times\mathbb{Z}$ with $|p| \leq \ell$ such that

$\Upsilon = \Upsilon_{\nu,p} = \Upsilon_{-\nu,-p}$ (the character of $\mathcal{Z}(\mathfrak{g})$ given by (1.3.3)); furthermore, for any $(\nu,p) \in \mathbb{C}\times\mathbb{Z}$ and all $\ell, q \in \mathbb{Z}$ with $\ell \geq |p|, |q|$, one has

$$\dim_{\mathbb{C}} W_0^{\mathrm{pol}}(\Upsilon_{\nu,p};\ell,q) = \dim_{\mathbb{C}} W_0(\Upsilon_{\nu,p};\ell,q) = 2 \tag{1.4.10}$$

and, when $0 \neq \omega \in \mathfrak{O}$,

$$\dim_{\mathbb{C}} W_\omega^{\mathrm{pol}}(\Upsilon_{\nu,p};\ell,q) \leq 1 \quad \text{and} \quad \dim_{\mathbb{C}} W_\omega(\Upsilon_{\nu,p};\ell,q) \leq 2\ , \tag{1.4.11}$$

with any generator, h, of $W_\omega^{\mathrm{pol}}(\Upsilon_{\nu,p};\ell,q)$ necessarily satisfying

$$h(na[r]k) \ll_h r^{\ell+1/2}\exp(-2\pi|\omega|r) \quad \text{for } n\in N, k\in K \text{ and } r\geq 1\ . \tag{1.4.12}$$

Given (1.4.1) and the remark ending with (1.4.8), the above shows too that $A_\Gamma(\Upsilon;\ell,q) = \{0\}$ unless it is the case that, for some $\nu\in\mathbb{C}$ and some integer $p\in[-\ell,\ell]$, one has $\Upsilon = \Upsilon_{\nu,p}$ (i.e. as in (1.3.3)): one then designates (ν,p) as the 'spectral parameters' of the space $A_\Gamma(\Upsilon;\ell,q)$ and its elements. Moreover, as is shown in Lemma 5.2.1 of [32], it follows from (1.4.12) and the remark ending with (1.4.9) that any cusp form $f\in A_\Gamma^0(\Upsilon_{\nu,p};\ell,q)$ must satisfy, for each cusp $\mathfrak{c}$ of Γ,

$$f(g_{\mathfrak{c}}na[r]k) \ll_{f,\mathfrak{c}} r^{\ell+1/2}\exp(-\pi r) \quad \text{for } n\in N, k\in K \text{ and } r\geq 1 \tag{1.4.13}$$

(note that the implicit constant here does not depend also upon $g_{\mathfrak{c}}$, since any alternative to $g_{\mathfrak{c}}$ permitted by (1.1.16) and (1.1.20)-(1.1.21) must lie in some coset $g_{\mathfrak{c}}h[\eta]N$ with $\eta^8 = 1$). By (1.4.13), (1.1.5)-(1.1.7), (1.1.14), (1.1.15) and (1.2.2) all cusp forms lie not only in $C^\infty(\Gamma\backslash G)$, but also in $L^2(\Gamma\backslash G)$ (as do all constant functions $f: G\rightarrow\mathbb{C}$). Another implication of (1.4.13) is that if $f: G\rightarrow\mathbb{C}$ is a non-zero constant function, then f is not a cusp form.

By (1.1.8), (1.1.9), (1.2.10), (1.2.11) and (1.3.1), the space $A_\Gamma(\Upsilon;\ell,q)$ may contain non-zero K-trivial functions only if $\ell = q = 0$, which (by the first observation of the previous paragraph) requires that $\Upsilon = \Upsilon_{\nu,p}$ for some $(\nu,p)\in\mathbb{C}\times\{0\}$. Conversely, by (1.3.5), all even functions $f\in C^\infty(G)$ of K-type $(0,0)$ are K-trivial. Suppose now that $f\in A_\Gamma^0(\Upsilon_{\nu,0};0,0)$ (i.e. that f is some K-trivial cusp form). Then, by the remarks preceding (1.2.13), one may treat f as an element of $L^2(\Gamma\backslash\mathbb{H}_3)$, and hence make (1.2.14), (1.2.16), (1.2.17) and (1.2.18)-(1.2.20) applicable to f. By (1.2.13) in particular, and (1.4.4), (1.4.5), (1.4.6), (1.2.8) and (1.3.3), one has, for $\nu\in\mathbb{C}$ and $f\in A_\Gamma^0(\Upsilon_{\nu,0};0,0)$,

$$-\boldsymbol{\Delta} f = \lambda_\nu f \quad \text{with} \quad \lambda_\nu = 1-\nu^2\ . \tag{1.4.14}$$

By (1.2.20) one must have $\lambda_\nu > 0$ in (1.4.14) when f is non-constant: given that cusp forms (excepting 0) are non-constant, this shows that $A_\Gamma^0(\Upsilon_{\nu,0};0,0)\neq\{0\}$ only if $\nu^2 < 1$. Consequently two cases may be distinguished: that of the 'principal series', in which $\nu^2\leq 0$ (so that $\lambda_\nu\geq 1$); and that of the 'complementary series', where one has $0<\nu^2<1$ (so that $0<\lambda_\nu<1$). If the generalised Selberg conjecture is true, then the complementary series is absent when (as is the case here) the relevant discrete group Γ is a congruence subgroup of $SL(2,\mathfrak{O})$. Though this conjecture remains open, the work [26] and [25], Theorem 4.10, of Kim and Shahidi has shown that

$$A_\Gamma^0(\Upsilon_{\nu,0};0,0)\neq 0 \quad \text{only if} \quad \nu^2\leq(2/9)^2\ , \tag{1.4.15}$$

so that in (1.4.14) one always has either $\lambda_\nu = 0$ or $\lambda_\nu\geq 77/81$. By (1.4.14) and (1.2.19) (and (1.2.5), (1.2.2) and (1.1.14)), the spaces $A_\Gamma^0(\Upsilon_{\nu,0};0,0), A_\Gamma^0(\Upsilon_{\xi,0};0,0)\subset L^2(\Gamma\backslash G)$ are mutually orthogonal if $\xi^2\neq\nu^2$. Similarly, since $-\boldsymbol{\Delta}\varphi_{0,0}(-1,0) = 0\varphi_{0,0}(-1,0)$ (whereas $\lambda_\nu>0$ in (1.4.14) if $f\in A_\Gamma^0(\Upsilon_{\nu,0};0,0)-\{0\}$) any non-zero K-trivial cusp form is orthogonal to the subspace $\mathbb{C}\varphi_{0,0}(-1,0)\subset L^2(\Gamma\backslash G)$ of constant functions; the same conclusion is true in respect of all cusp forms (K-trivial or not): for if there is a cusp form f of K-type (ℓ,q) that is not K-trivial, then ℓ must be a positive integer, and so the conclusion that $\langle f,\varphi_{0,0}(-1,0)\rangle_{\Gamma\backslash G} = 0$ follows by (1.2.2), (1.1.14) and both (1.3.5) and the observations on $\mathcal{L}_{\mathrm{even}}(K)$ preceding it.

§1.5 The Jacquet integral; generalised Kloosterman sums and the Fourier expansion of a Poincaré series; Fourier coefficients of cusp forms.

In the case where f is a cusp form, the Fourier expansion (1.4.1) may be made much more explicit if one has, for each non-zero $\omega \in \mathfrak{O}$, an explicitly defined non-zero function in the space $W_\omega^{\mathrm{pol}}(\Upsilon_{\nu,p};\ell,q)$. The key to this will be the Jacquet integral, $\mathbf{J}_\omega f : G \mapsto \mathbb{C}$, which for $\omega \in \mathbb{C}$ and functions $f \in C^\infty(G)$ that, for some $\sigma > 0$ and some $r_1 > 0$, satisfy

$$f(na[r]k) \ll r^{1+\sigma} \quad \text{for } n \in N,\ k \in K \text{ and } 0 < r \le r_1\,, \tag{1.5.1}$$

is given by

$$(\mathbf{J}_\omega f)(g) = \int_N (\psi_\omega(n))^{-1} f(k[0,-1]ng)\,\mathrm{d}n \quad \text{for } g \in G\,. \tag{1.5.2}$$

By this definition, and by (1.2.6), it follows that if σ is a positive real number, if $f \in C^\infty(G)$ satisfies (1.5.1), and if $\mathbf{X} \in \mathfrak{sl}(2,\mathbb{C})$ is such that (1.5.1) remains valid following the substitution of $\mathbf{X}f$ for f, then one has

$$(\mathbf{X}\mathbf{J}_\omega f)(g) = (\mathbf{J}_\omega \mathbf{X} f)(g) \qquad (g \in G). \tag{1.5.3}$$

The Jacquet integral arises naturally, via the Bruhat decomposition $\Gamma = \Gamma_\infty \sqcup (\Gamma \cap Pk[0,-1]N)$, in connection with the Fourier expansions of certain Poincaré series (see Section 5 of [5] for a concise sketch of the details). A digression on Poincaré series and their Fourier expansions now facilitates the introduction of other important concepts and terminology: discussion of the Jacquet integral as it relates to $W_\omega^{\mathrm{pol}}(\Upsilon_{\nu,p};\ell,q)$ resumes after that.

Let $\mathfrak{a}$ be a cusp of Γ. Then, for suitable functions f lying in the space

$$C^\infty(B^+\backslash G) = \left\{\phi \in C^\infty(G) : \phi(bg) = \phi(g) \text{ for } b \in B^+,\ g \in G\right\}$$

(where B^+ is as in (1.1.20)-(1.1.21)), one has a Poincaré series, $P^{\mathfrak{a}} f \in C^\infty(\Gamma\backslash G)$, given by

$$(P^{\mathfrak{a}} f)(g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma} f\left(g_{\mathfrak{a}}^{-1}\gamma g\right) \quad \text{for } g \in G\,. \tag{1.5.4}$$

In Subsection 6.2 (below) we describe criteria with the aid of which we are able to establish, ultimately in Subsection 6.5, that certain non-constant functions f are 'suitable' (in the above sense).

Supposing that $\omega, \omega' \in \mathfrak{O}$, take $f = f_\omega$ to be a suitable function in the space $C^\infty(N\backslash G,\omega) \subset C^\infty(B^+\backslash G)$ given by (1.4.7) and (1.4.3). Then via the Bruhat decomposition of $\Gamma'_{\mathfrak{a}}\backslash\Gamma$ one can ascertain that, in the Fourier expansion of $(P^{\mathfrak{a}} f_\omega)(g)$ at an arbitrary cusp $\mathfrak{a}'$ of Γ, the Fourier term of order ω' is

$$\begin{aligned}\left(F_{\omega'}^{\mathfrak{a}'} P^{\mathfrak{a}} f_\omega\right)(g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} &\sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma\ :\ \gamma\mathfrak{a}'=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{a}'} \in h[u(\gamma)]N}} f_\omega\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{a}'} g\right) \delta_{\omega u(\gamma),\omega'/u(\gamma)} \,+ \\ &+ \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}'}} S_{\mathfrak{a},\mathfrak{a}'}(\omega,\omega';c) \left(\mathbf{J}_{\omega'} \mathbf{h}_{1/c} f_\omega\right)(g)\,,\end{aligned} \tag{1.5.5}$$

where

$$\delta_{\rho,\sigma} = \begin{cases} 1 & \text{if } \rho = \sigma, \\ 0 & \text{otherwise,}\end{cases} \tag{1.5.6}$$

and

$$\mathbf{h}_u f(g) = f(h[u]g) \quad \text{for } g \in G\,, \tag{1.5.7}$$

and where, with

$${}^{\mathfrak{a}}\Gamma^{\mathfrak{a}'}(c) = \left\{\gamma \in \Gamma : g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{a}'} = \begin{pmatrix} * & * \\ c & * \end{pmatrix}\right\} \quad \text{for } c \in \mathbb{C}\,, \tag{1.5.8}$$

one has

$$ {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}'} = \left\{ c \in \mathbb{C} - \{0\} : {}^{\mathfrak{a}}\Gamma^{\mathfrak{a}'}(c) \neq \emptyset \right\} \tag{1.5.9} $$

and, for $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}'}$,

$$ S_{\mathfrak{a},\mathfrak{a}'}(\omega, \omega'; c) = \sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{a}'}(c) / \Gamma'_{\mathfrak{a}'} \\ g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}'} = \begin{pmatrix} s(\gamma) & * \\ c & d(\gamma) \end{pmatrix}}} \mathrm{e}\left(\mathrm{Re}\left(\omega \, \frac{s(\gamma)}{c} + \omega' \, \frac{d(\gamma)}{c} \right) \right). \tag{1.5.10} $$

Given the restricted choice of scaling matrices $g_{\mathfrak{a}}, g_{\mathfrak{a}'}$ (as in (1.1.16) and (1.1.20)-(1.1.21)), one can show that

$$ {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}'} \ni c \quad \text{only if} \quad 0 \neq c^2 \in \mathfrak{O} \;\text{ and }\; c^2 \sim (c')^2 \, m_{\mathfrak{a}} m_{\mathfrak{a}'} \;\text{ for some }\; c' \in \mathfrak{O} \tag{1.5.11} $$

(where the non-zero Gaussian integers $m_{\mathfrak{c}}$ are as defined just below (1.1.22)). For $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}'}$ and c' as in (1.5.11), the 'generalised Kloosterman sum' in (1.5.10) trivially satisfies $|S_{\mathfrak{a},\mathfrak{a}'}(\omega, \omega'; c)| \leq |m_{\mathfrak{a}} m_{\mathfrak{a}'} c'|^2 = |c|^2 \, |m_{\mathfrak{a}} m_{\mathfrak{a}'}|$, while work of Bruggeman and Miatello in Proposition 9 and Theorem 10 of [4] shows, by means of an exponential sum estimate of A. Weil, that one has the generally non-trivial bound:

$$ |S_{\mathfrak{a},\mathfrak{a}'}(\omega, \omega'; c)| \leq \sqrt{8} \, |m_{\mathfrak{a}} m_{\mathfrak{a}'}|^2 \, |(c', q_0^{\infty}) \, (c', \omega, \omega') \, c'| \, \tau(c'), \tag{1.5.12} $$

where $|(c', q_0^{\infty})| = \lim_{n\to\infty} |(c', q_0^n)|$ and $\tau(c')$ is the number of Gaussian integer divisors of c'. The generalised Kloosterman sums also satisfy some symmetry relations:

$$ S_{\mathfrak{a},\mathfrak{a}'}(\omega, \omega'; c) = S_{\mathfrak{a}',\mathfrak{a}}(-\omega', -\omega; -c) = S_{\mathfrak{a}',\mathfrak{a}}(-\omega', -\omega; c) \tag{1.5.13} $$

(the first of these equations following from the observation that if $\{\gamma_1, \ldots, \gamma_n\}$ is a complete set of representatives for the set of double cosets $\Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{a}'}(c)/\Gamma'_{\mathfrak{a}'}$, then $\left\{\gamma_1^{-1}, \ldots, \gamma_n^{-1}\right\}$ is a complete set of representatives for $\Gamma'_{\mathfrak{a}'} \backslash {}^{\mathfrak{a}'}\Gamma^{\mathfrak{a}}(-c)/\Gamma'_{\mathfrak{a}}$; the second following since ${}^{\mathfrak{a}'}\Gamma^{\mathfrak{a}}(-c) = h[-1] {}^{\mathfrak{a}'}\Gamma^{\mathfrak{a}}(c)$).

Returning now to the subject of the Jacquet integral's rôle in providing an explicit generator for the space $W_{\omega}^{\mathrm{pol}}(\Upsilon_{\nu,p}; \ell, q)$, suppose that $\nu \in \mathbb{C}$ and $p, q, \ell \in \mathbb{Z}$ with $|p|, |q| \leq \ell$. Then it can be verified that the function $\varphi_{\ell,q}(\nu, p) : G \to \mathbb{C}$ (defined in (1.3.2)) lies in the space $W_0^{\mathrm{pol}}(\Upsilon_{\nu,p}; \ell, q)$; and that, given (1.4.10), it is furthermore the case that

$$ W_0(\Upsilon_{\nu,p}; \ell, q) = \mathbb{C} \, \varphi_{\ell,q}(\nu, p) \oplus \begin{cases} \mathbb{C} \dfrac{\partial}{\partial \nu} \, \varphi_{\ell,q}(\nu, 0) \Big|_{\nu=0} & \text{if } \nu = p = 0, \\ \mathbb{C} \, \varphi_{\ell,q}(-\nu, -p) & \text{otherwise.} \end{cases} \tag{1.5.14} $$

In addition, since (1.5.3) implies that the Jacquet integral has, in common with the Fourier operators $F_{\omega}^{\mathfrak{c}}$, the property of commuting with the actions (as differentiable operators) of all elements of $\mathcal{U}(\mathfrak{g})$ (commuting, in particular, with $\Omega_{\pm}$ and the extensions of $\Omega_{\mathfrak{k}}$ and $\mathbf{H}_2$), it therefore follows by (1.3.2), (1.5.12), (1.5.1)-(1.5.2) and the alternative representation of $\mathbf{J}_{\omega} \varphi_{\ell,q}(\nu, p)$ in Equation (5.10) of [5], that one has

$$ W_{\omega}^{\mathrm{pol}}(\Upsilon_{\nu,p}; \ell, q) \ni \mathbf{J}_{\omega} \varphi_{\ell,q}(\nu, p) \quad \text{for } \omega \in \mathbb{C}, \text{ if } \mathrm{Re}(\nu) > 0\,. \tag{1.5.15} $$

The Jacquet integral $(\mathbf{J}_{\omega} \varphi_{\ell,q}(\nu, p))(g)$ (where $g \in G$) fails to converge absolutely when $\mathrm{Re}(\nu) \leq 0$. It nevertheless follows from Lemma 5.1 of [5] that when $\omega \neq 0$ the holomorphic function $\nu \mapsto (\mathbf{J}_{\omega} \varphi_{\ell,q}(\nu, p))(g)$ (with domain $\{\nu \in \mathbb{C} : \mathrm{Re}(\nu) > 0\}$) has an extension via analytic continuation that is entire; and that, for each $\nu \in \mathbb{C}$, the resulting function $\mathbf{J}_{\omega} \varphi_{\ell,q}(\nu, p) : G \to \mathbb{C}$ is an element of the space $W_{\omega}^{\mathrm{pol}}(\Upsilon_{\nu,p}; \ell, q)$ distinct from 0. Given this extension of (1.5.15), it follows by (1.4.11) that

$$ W_{\omega}^{\mathrm{pol}}(\Upsilon_{\nu,p}; \ell, q) = \mathbb{C} \, \mathbf{J}_{\omega} \varphi_{\ell,q}(\nu, p) \quad \text{for } \omega \neq 0\,. \tag{1.5.16} $$

By this and the remarks encompassing (1.4.9) it moreover follows that if $\mathfrak{c}$ is a cusp of Γ, $f \in A_{\Gamma}^{\mathrm{pol}}(\Upsilon_{\nu,p};\ell,q)$ and $0 \neq \omega \in \mathfrak{O}$ then

$$\left(F_{\omega}^{\mathfrak{c}} f\right)(g) = c_f^{\mathfrak{c}}(\omega)\left(\mathbf{J}_{\omega}\varphi_{\ell,q}(\nu,p)\right)(g) \quad \text{for } g \in G\,, \tag{1.5.17}$$

where $c_f^{\mathfrak{c}}(\omega)$ (the Fourier coefficient) is a complex number depending only upon ω, $g_{\mathfrak{c}}$ and f.

In the case of $(\mathbf{J}_0\varphi_{\ell,q}(\nu,p))(g)$ the extension by analytic continuation with respect to $\nu \in \mathbb{C}$ is meromorphic on $\mathbb{C}$, and is given by

$$\mathbf{J}_0\varphi_{\ell,q}(\nu,p) = \pi\,\frac{\Gamma(\ell+1-\nu)\Gamma(|p|+\nu)}{\Gamma(\ell+1+\nu)\Gamma(|p|+1-\nu)}\,\varphi_{\ell,q}(-\nu,-p) \tag{1.5.18}$$

whenever the right-hand side is defined.

§1.6 The spaces $H(\nu,p)$ of K-finite functions; principal and complementary series.

The results in this paper concern sums involving the Fourier coefficients (as given by (1.5.17)) of orthogonal systems of cusp forms. A significant aid in describing the relevant systems of cusp forms are certain spaces $H(\nu,p) \subset C^{\infty}(N\backslash G)$ that have the functions $\varphi_{\ell,q}(\nu,p)$ as their generators. To motivate the definition of $H(\nu,p)$ (which follows) one may first observe that the $(2\ell+1)\times(2\ell+1)$ matrices $\mathbf{\Phi}_{\ell} = \left(\Phi_{p,q}^{\ell}\right)$ satisfy

$$\mathbf{\Phi}_{\ell}(k_1k_2) = \mathbf{\Phi}_{\ell}(k_1)\,\mathbf{\Phi}_{\ell}(k_2) \quad \text{for } k_1,k_2 \in K \text{ and } \ell = 0,1,2,\ldots$$

(a slight change in the definition of $\mathbf{\Phi}_{\ell}$, involving a normalisation of the entries $\Phi_{p,q}^{\ell}$, would in fact transform it into a realisation of the unitary representation of degree $2\ell+1$ of K). The functions $\varphi_{\ell,q}(\nu,p) : G \mapsto \mathbb{C}$ are therefore K-finite (each set of 'K-translates' $\{g \mapsto \varphi_{\ell,q}(\nu,p)(gk) : k \in \mathbb{K}\}$ being contained in the span of the finite set $\{\varphi_{\ell,q'}(\nu,p) : q' \in \mathbb{Z} \text{ and } |q'| \leq \ell\}$). Therefore, for each of the pairs $(\nu,p) \in ((i\mathbb{R})\times\mathbb{Z}) \cup ((-1,1)\times\{0\})$ (in particular), one has the space

$$H(\nu,p) = \{\text{finite linear combinations of functions } \varphi_{\ell,q}(\nu,p) \text{ with } \ell,q \in \mathbb{Z} \text{ and } \ell \geq |q|,|p|\}\ , \tag{1.6.1}$$

which contains only K-finite functions. Though $H(\nu,p)$ is invariant under the action of K by right translation, it is not so under the action of G by right translation. Nevertheless, these spaces $H(\nu,p)$ are $\mathfrak{g}$-invariant, and each constitutes an irreducible representation space for $\mathfrak{g}$.

If one excludes the case in which $\omega = 0$ and $\nu = p = 0$, then the extension by analytic continuation of the Jacquet integral $(\mathbf{J}_{\omega}\varphi_{\ell,q}(\nu,p))(g)$ (see between (1.5.15) and (1.5.18)) permits a unique further extension to a linear operator,

$$\mathbf{J}_{\omega}^{\nu,p} : H(\nu,p) \to W_{\omega}^{\mathrm{pol}}(\Upsilon_{\nu,p}) = \bigoplus_{\ell=|p|}^{\infty}\bigoplus_{q=-\ell}^{\ell} W_{\omega}^{\mathrm{pol}}(\Upsilon_{\nu,p};\ell,q)$$

(the 'Jacquet operator'): note the relevance of (1.5.14) and (1.5.18) for the case $\omega = 0$. The form of the analytic continuation provided by Lemma 5.1 of [5] is such as to ensure that the Jacquet operator $\mathbf{J}_{\omega}^{\nu,p}$ does (even when $\mathrm{Re}(\nu) \leq 0$) inherit from the Jacquet integral $(\mathbf{J}_{\omega}f)(g)$ the property of commuting with the actions of all $\Psi \in \mathcal{U}(\mathfrak{g})$.

Let $H^{\infty}(\nu,p)$ be the space of functions $f \in C^{\infty}(N\backslash G)$ that satisfy $f\big(a[r]h\big[e^{it}\big]k\big) = r^{1+\nu}e^{-2pit}f(k)$ for $r > 0$, $t \in \mathbb{R}$ and $k \in K$. The inner-product appropriate for $H(\nu,p)$ derives from the duality between the pair of spaces $H^{\infty}(\pm\nu,\pm p)$ given by the bilinear form

$$\langle f_+, f_-\rangle_{\natural} = \int_K f_+(k)f_-(k)\,\mathrm{d}k \quad (f_{\pm} \in H^{\infty}(\pm\nu,\pm p)). \tag{1.6.2}$$

One has $H^{\infty}(\nu,p) \supset H(\nu,p)$ and $H^{\infty}(-\nu,-p) \supset H(-\nu,-p)$; and if $(\nu,p) \in (i\mathbb{R})\times\mathbb{Z}$ (which is the 'unitary principal series' case), then $\overline{f} \in H(\overline{\nu},-p) = H(-\nu,-p)$ when $f \in H(\nu,p)$, so that one may define the inner product:

$$(f_1,f_2)_{\mathrm{ps}} = \left\langle f_1,\overline{f_2}\right\rangle_{\natural} \quad \text{for } f_1,f_2 \in H(\nu,p) \text{ and } (\nu,p) \in (i\mathbb{R})\times\mathbb{Z}\,. \tag{1.6.3}$$

If instead $p = 0$ and $0 < \nu^2 < 1$ (the 'complementary series' case), then one has $\overline{f} \in H(\overline{\nu}, 0) = H(\nu, 0)$ when $f \in H(\nu, p)$. Then, in order to make applicable the duality of (1.6.2), one passes from $\overline{f} \in H(\nu, 0)$ to $\pi^{-1}(\Gamma(1-\nu)/\Gamma(\nu))\mathbf{J}_0^{\nu,0}\overline{f} \in H(-\nu, 0)$. Since $\mathbf{J}_0^{\nu,0}\overline{f} = \overline{\mathbf{J}_0^{\nu,0} f}$, this leads to the definition:

$$(f_1, f_2)_{cs} = \frac{\Gamma(1-\nu)}{\pi\Gamma(\nu)} \left\langle f_1, \overline{\mathbf{J}_0^{\nu,0} f_2} \right\rangle_{\natural} \quad \text{for } f_1, f_2 \in H(\nu, 0) \text{ with } 0 < \nu^2 < 1 \,. \tag{1.6.4}$$

Since $\mathbf{J}_0^{\nu,0}$ is a linear operator that commutes with the actions of the elements of $\mathcal{U}(\mathfrak{g})$ upon the space $H(\nu, 0)$, and since $\langle f_+, f_- \rangle_\natural$ is invariant under any right translation by an element of G (applied simultaneously to $f_\pm \in H^\infty(\pm\nu, 0)$), completion of $H(\nu, 0)$ with respect to the norm $\|f\|_{\rm cs} = \sqrt{(f,f)_{\rm cs}}$ yields a Hilbert space $H^2(\nu, 0) \supset H^\infty(\nu, 0)$ upon which G acts unitarily (as G also does in the 'principal series' case, where $H^2(\nu, p) \supset H^\infty(\nu, p)$ is instead the completion of $H(\nu, p)$ with respect to the norm $\|f\|_{\rm ps} = \sqrt{(f,f)_{\rm ps}}$). See [32], Section 2.3, for a fuller discussion of the spaces $H^\infty(\nu, p)$ and $H^2(\nu, p)$.

Focusing on the generators of $H(\nu, p)$, one finds by (1.6.4) and (1.5.18) that, for $p = 0$ and $0 < \nu^2 < 1$,

$$\begin{aligned} (\varphi_{\ell,q}(\nu,0), \varphi_{\ell',q'}(\nu,0))_{\rm cs} &= \left(\Phi^{\ell}_{0,q} \,, \frac{\Gamma(1+\ell'-\nu)}{\Gamma(1+\ell'+\nu)} \, \Phi^{\ell'}_{0,q'} \right)_K = \\ &= \delta_{\ell,\ell'}\delta_{q,q'} \frac{\Gamma(1+\ell-\nu)}{\Gamma(1+\ell+\nu)} \frac{1}{(\ell+\frac{1}{2})} \binom{2\ell}{\ell} \binom{2\ell}{\ell-q}^{-1} \,, \end{aligned} \tag{1.6.5}$$

while, for $(\nu, p) \in (i\mathbb{R}) \times \mathbb{Z}$ with $|p| \leq \ell$,

$$(\varphi_{\ell,q}(\nu,p), \varphi_{\ell',q'}(\nu,p))_{\rm ps} = \left(\Phi^{\ell}_{p,q} \,, \Phi^{\ell'}_{p,q'}\right)_K = \delta_{\ell,\ell'}\delta_{q,q'} \frac{1}{(\ell+\frac{1}{2})} \binom{2\ell}{\ell-p} \binom{2\ell}{\ell-q}^{-1} \,. \tag{1.6.6}$$

Note that by (1.6.1), (1.6.5) and (1.6.6), it is evident that the inner products $(\cdot,\cdot)_{\rm ps}$ and $(\cdot,\cdot)_{\rm cs}$ are indeed positive definite on the relevant spaces $H(\nu, p)$.

Since the spaces $H(\nu, p)$ and $H^\infty(\nu, p)$ have been defined only for integer values of p (and given their definitions, including (1.3.2), along with the points noted in connection with (1.3.4) and (1.3.5)) it is implied that these spaces contain only even functions. The same is therefore true of the completed spaces $H^2(\nu, p)$.

§1.7 Decomposing the space $L^2(\Gamma\backslash G)$.

The utility of the spaces $H(\nu, p)$ in studying cusp forms derives from their rôle in classifying the irreducible unitary representations of $SL(2, \mathbb{C})$. More specifically, it is known that any even non-trivial irreducible unitary representation of the Lie group $G = SL(2, \mathbb{C})$ is, for some $(\nu, p) \in ((i\mathbb{R}) \times \mathbb{Z}) \cup ((-1, 1) \times \{0\})$, unitarily equivalent to a certain representation of G with representation space $H^2(\nu, p)$; this representation, being (of necessity) unitary itself, is a strongly continuous homomorphism,

$$P^{2p,2\nu} : G \to U\big(H^2(\nu, p)\big) \ , \tag{1.7.1}$$

mapping elements $g \in G$ to elements $P_g^{2p,2\nu}$ of the group $U\big(H^2(\nu, p)\big)$ of unitary operators on $H^2(\nu, p)$. For all $g \in G$, the defining property of the operator $P_g^{2p,2\nu} : H^2(\nu, p) \to H^2(\nu, p)$ is that

$$\left(P_g^{2p,2\nu}\varphi\right)(h) = \varphi(hg) \quad \text{for } \varphi \in H^2(\nu, p) \text{ and } h \in G\,. \tag{1.7.2}$$

Note that this is the so-called 'induced picture' of the classification: Theorem 16.2 of [27] gives the 'non-compact' picture (without the restriction to even representations) and Section 7.1 [27] describes the 'induced', 'compact' and 'non-compact' pictures in relation to one another.

The above classification is significant in the current context, since (as already noted) each cusp form f lies in $L^2(\Gamma\backslash G)$, and is orthogonal to the subspace of constant functions. Hence, on identifying the constant functions with elements of $\mathbb{C}$, one has:

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus {}^0L^2(\Gamma\backslash G) \oplus {}^{\rm e}L^2(\Gamma\backslash G) \ , \tag{1.7.3}$$

where ${}^{0}L^2(\Gamma\backslash G)$ is the closure of the space spanned by the set of all cusp forms (of arbitrary K-type (ℓ,q), and with any character $\Upsilon_{\nu,p}$) and ${}^{\mathrm{e}}L^2(\Gamma\backslash G)$ is the orthogonal complement of $\mathbb{C}\oplus{}^{0}L^2(\Gamma\backslash G)$ in $L^2(\Gamma\backslash G)$. The space ${}^{0}L^2(\Gamma\backslash G)$ is invariant with respect to the right-actions of the elements of G, and one has

$$ {}^{0}L^2(\Gamma\backslash G)=\overline{\bigoplus V}\ , \tag{1.7.4}$$

where the direct sum is that of countably many pairwise orthogonal infinite-dimensional closed subspaces V, each of which is invariant and irreducible with respect to the right-actions of the elements of G. Since all functions in the space $L^2(\Gamma\backslash G)$ are (by the observation following (1.2.1)) necessarily functions that are even, so too are all functions lying in ${}^{0}L^2(\Gamma\backslash G)$, or ${}^{\mathrm{e}}L^2(\Gamma\backslash G)$, or in any one of the above factors V.

Now (1.2.2) and the unimodularity of the Haar measure on G imply that

$$\int_{\Gamma\backslash G} f\,(gm)\,\overline{h\,(gm)}\,\mathrm{d}g=\int_{\Gamma\backslash G} f\,(g)\,\overline{h\,(g)}\,\mathrm{d}g=\langle f,h\rangle_{\Gamma\backslash G}\quad\text{for } f,h\in{}^{0}L^2(\Gamma\backslash G)\text{ and } m\in G\,. \tag{1.7.5}$$

One has therefore (for each V in (1.7.4)) the non-trivial irreducible unitary representation $R^V:G\to U(V)$, which for $m\in G$ maps m to the right-action R^V_m that has $\left(R^V_m f\right)(g)=f(gm)$ when $f\in V$, $g\in G$; and, as all functions in the space V are even, this representation R^V is also even (i.e. $R^V_{h[-1]}=R^V_{h[1]}$). By the discussion, around (1.7.1), (1.7.2), concerning the classification of such representations of G it follows that, for each V in (1.7.4), there exists $(\nu_V,p_V)\in((i\mathbb{R})\times\mathbb{Z})\cup((-1,1)\times\{0\})$ and a surjective linear isometry $\tilde{T}_V:H^2\,(\nu_V,p_V)\to V$ such that

$$R^V_g\tilde{T}_V=\tilde{T}_V P_g^{2p_V,2\nu_V}\quad\text{for } g\in G\,. \tag{1.7.6}$$

The operator $T_V=\tilde{T}_V\big|_{H(\nu_V,p_V)}$ has a dense image $V_K\subset V$ (the K-finite subspace of V). Hence (and since T_V is an isometry) one has by (1.6.1), (1.7.4) and the relevant orthogonality relations, (1.6.5) or (1.6.6), the decomposition:

$$V_K=\bigoplus_{\ell=|p_V|}^{\infty}\ \bigoplus_{q=-\ell}^{\ell} V_{K,\ell,q}\subset V\subset{}^{0}L^2(\Gamma\backslash G)\ , \tag{1.7.7}$$

where

$$V_{K,\ell,q}=\mathbb{C}T_V\varphi_{\ell,q}\,(\nu_V,p_V)\ . \tag{1.7.8}$$

Though T_V does not inherit from $\tilde{T}_V$ the property of commuting (as in (1.7.6)) with the right-actions of all elements of G, it does nevertheless follow from (1.7.6) that

$$\mathbf{X}T_V=T_V\mathbf{X}\quad\text{for }\mathbf{X}\in\mathfrak{sl}(2,\mathbb{C})\,. \tag{1.7.9}$$

Therefore T_V commutes with the actions, as differential operators, of all elements of $\mathcal{U}(\mathfrak{g})$.

By (1.7.9), the operator T_V shares with the Jacquet operators $\mathbf{J}_\omega^{\nu,p}$ the property of preserving the K-type and character Υ of the functions on which it operates. Therefore, if one supposes now that $\ell,q\in\mathbb{Z}$, $\ell\geq q$ and $\ell\geq|p_V|$, then it follows by (1.7.7), (1.7.8), (1.5.13) and (1.4.8) that $T_V\varphi_{\ell,q}\,(\nu_V,p_V)\in A_\Gamma\,(\Upsilon_{\nu_V,p_V};\ell,q)\cap{}^{0}L^2(\Gamma\backslash G)$. This in fact implies that

$$V_{K,\ell,q}\subseteq A^0_\Gamma\,(\Upsilon_{\nu_V,p_V};\ell,q)\ . \tag{1.7.10}$$

A proof of (1.7.10) may be given along the following lines. Let $\mathfrak{c}$ be any cusp of Γ. Then by (1.4.1)-(1.4.2) (and since $T_V\varphi_{\ell,q}\,(\nu_V,p_V)\in A_\Gamma\,(\Upsilon_{\nu_V,p_V};\ell,q)\subset C^\infty(\Gamma\backslash G)$), one has the Fourier expansion:

$$\left(T_V\varphi_{\ell,q}\,(\nu_V,p_V)\right)(g_{\mathfrak{c}}g)=\sum_{\omega\in\mathfrak{O}}\left(F^{\mathfrak{c}}_\omega T_V\varphi_{\ell,q}\,(\nu_V,p_V)\right)(g)\qquad(g\in G), \tag{1.7.11}$$

where, by the point noted in connection with (1.4.8), $F^{\mathfrak{c}}_\omega T_V\varphi_{\ell,q}\,(\nu_V,p_V)\in W_\omega\,(\Upsilon_{\nu_V,p_V};\ell,q)$ for $\omega\in\mathfrak{O}$. Since $T_V\varphi_{\ell,q}\,(\nu_V,p_V)\in{}^{0}L^2(\Gamma\backslash G)$, each term in this Fourier expansion is necessarily square integrable over $g_{\mathfrak{c}}^{-1}\mathcal{E}_{\mathfrak{c}}K^+$, where $\mathcal{E}_{\mathfrak{c}}K^+\subset G$ is the 'cusp sector' defined by (1.1.14) and (1.1.22)-(1.1.23) (to prove this one uses (1.4.7), (1.4.8) and the fact that the characters ψ_ω defined in (1.4.3) satisfy $\int_{B^+\backslash N}\psi_{\omega'}(n)\overline{\psi_\omega(n)}\mathrm{d}n=0$

when ω and ω' are distinct Gaussian integers). In respect of the particular case $\omega = 0$, it moreover follows that, since cusp forms span a dense subspace of ${}^0L^2(\Gamma\backslash G)$, one must have:

$$0 = \int_{g_{\mathfrak{c}}^{-1}\mathcal{E}_{\mathfrak{c}}K^+} \left|\left(F_0^{\mathfrak{c}} T_V \varphi_{\ell,q}\left(\nu_V, p_V\right)\right)(g)\right|^2 \mathrm{d}g = \frac{1}{[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]} \int_{|m_{\mathfrak{c}}|^{-1}}^{\infty} \int_K \left|\left(F_0^{\mathfrak{c}} T_V \varphi_{\ell,q}\left(\nu_V, p_V\right)\right)(a[r]k)\right|^2 \mathrm{d}k r^{-3} \mathrm{d}r\ .$$

By combining these observations with the analysis of the space $W_0\left(\Upsilon_{\nu,p};\ell,q\right)$ worked out (via (1.3.5) and elements of the theory of Bessel functions) in the proof of Lemma 4.2 of [5], it may be deduced that

$$\left(F_0^{\mathfrak{c}} T_V \varphi_{\ell,q}\left(\nu_V, p_V\right)\right)(g) = 0 \quad \text{ for } g \in G\,, \tag{1.7.12}$$

and that $F_\omega^{\mathfrak{c}} T_V \varphi_{\ell,q}\left(\nu_V, p_V\right) \in W_\omega^{\mathrm{pol}}\left(\Upsilon_{\nu_V,p_V};\ell,q\right)$ for $0 \neq \omega \in \mathfrak{O}$. Therefore it follows by (1.5.16)-(1.5.17) that the Fourier expansion (1.7.11) has the special form:

$$\left(T_V \varphi_{\ell,q}\left(\nu_V, p_V\right)\right)(g_{\mathfrak{c}} g) = \sum_{0 \neq \omega \in \mathfrak{O}} c_V^{\mathfrak{c}}(\omega) \left(\mathbf{J}_\omega \varphi_{\ell,q}\left(\nu_V, p_V\right)\right)(g) \quad \text{ for } g \in G\,. \tag{1.7.13}$$

The Fourier coefficients $c_V^{\mathfrak{c}}(\omega)$ in (1.7.13) depend only upon V, T_V, $\mathfrak{c}$, $g_{\mathfrak{c}}$ and ω: for both $F_\omega^{\mathfrak{c}} T_V$ and $\mathbf{J}_\omega^{\nu_V,p_V}$ commute with all $\Psi \in \mathcal{U}(\mathfrak{g})$; and since $H\left(\nu_V, p_V\right)$ has the definition (1.6.1), and is irreducible with respect to the actions of the elements of $\mathcal{U}(\mathfrak{g})$, one must have $\Psi\varphi_{|p_V|,0}\left(\nu_V, p_V\right) = \varphi_{\ell,q}\left(\nu_V, p_V\right)$ for some $\Psi \in \mathcal{U}(\mathfrak{g})$. Finally, as somewhat of a converse to (1.5.17) and the results noted between (1.4.7) and (1.4.9), it follows by Lemma 5.2.1 of [32] that the existence of the expansion (1.7.13), when combined with the fact that $T_V \varphi_{\ell,q}\left(\nu_V, p_V\right) \in A_\Gamma\left(\Upsilon_{\nu_V,p_V};\ell,q\right)$, is sufficient to imply that the growth condition (1.4.13) is satisfied when $f = T_V \varphi_{\ell,q}\left(\nu_V, p_V\right)$. This (given the arbitrary choice of cusp $\mathfrak{c}$) shows that $T_V \varphi_{\ell,q}\left(\nu_V, p_V\right) \in A_\Gamma^{\mathrm{pol}}\left(\Upsilon_{\nu_V,p_V};\ell,q\right)$. Hence and by (1.7.12) and (1.7.8), one obtains (1.7.10). See [11], Theorem 3.3.1, for the corresponding proof in the K-trivial case.

As indicated by (1.7.10) and (1.7.7), the spectral parameters (ν_V, p_V) are shared by all elements of V_K (i.e. $\Omega_\pm f = \Upsilon_{\nu_V,p_V}\left(\Omega_\pm\right) f$ for $f \in V_K$ and either choice of sign, where the eigenvalues $\Upsilon_{\nu_V,p_V}\left(\Omega_\pm\right)$ are as given by (1.3.3)). Therefore (and since V_K is dense in V) one calls (ν_V, p_V) the 'spectral parameters of V'. By (1.7.8) and (1.7.10), the one-dimensional space $\mathbb{C}T_V \varphi_{\ell,q}\left(\nu_V, p_V\right)$ is the subspace $V_{K,\ell,q} \subset V_K$ spanned by all cusp-forms of K-type (ℓ, q) in V_K. With the operator T_V being an isometry, one necessarily has:

$$\left\|T_V \varphi_{\ell,q}\left(\nu_V, p_V\right)\right\|_{\Gamma\backslash G} = \begin{cases} \left\|\varphi_{\ell,q}\left(\nu_V, p_V\right)\right\|_{\mathrm{ps}} & \text{if } (\nu_V, p_V) \in (i\mathbb{R}) \times \mathbb{Z}\,, \\ \left\|\varphi_{\ell,q}\left(\nu_V, p_V\right)\right\|_{\mathrm{cs}} & \text{if } 0 < \nu_V^2 < 1 \text{ and } p_V = 0\,, \end{cases} \tag{1.7.14}$$

where the norms $\|\cdot\|_{\mathrm{ps}}$ and $\|\cdot\|_{\mathrm{cs}}$ are as described between (1.6.4) and (1.6.6). Therefore V, $\mathfrak{c}$ and $g_{\mathfrak{c}}$ determine both the set $\{e^{i\phi} T_V \varphi_{\ell,q}\left(\nu_V, p_V\right) : 0 \leq \phi < 2\pi\}$ and the set $\{(e^{i\phi} c_V^{\mathfrak{c}}(\omega))_{\omega \in \mathfrak{O}-\{0\}} : 0 \leq \phi < 2\pi\}$. It nevertheless later becomes convenient to work instead with modified Fourier coefficients:

$$C_V^{\mathfrak{c}}\left(\omega; \nu_V, p_V\right) = (\pi|\omega|)^{\nu_V} \left(\omega/|\omega|\right)^{-p_V} c_V^{\mathfrak{c}}(\omega). \tag{1.7.15}$$

By the point noted below (1.4.13), the dependence of $c_V^{\mathfrak{c}}(\omega)$ upon $g_{\mathfrak{c}}$ is quite simple. Indeed, the Fourier coefficients $c_V^{\mathfrak{c}}(\omega)$ are, as one would expect, essentially determined by the Γ-equivalence class of the cusp $\mathfrak{c}$: for if $\mathfrak{a} \overset{\Gamma}{\sim} \mathfrak{b}$ and $\tau \in \mathbb{C}$, $\eta \in \{u \in \mathbb{C} : u^8 = 1\}$ are such that $h[\eta]n[\tau] \in g_{\mathfrak{a}}^{-1}\Gamma g_{\mathfrak{b}}$ then

$$c_V^{\mathfrak{b}}(\omega) = \eta^{2p_V}\, \mathrm{e}\left(\mathrm{Re}(\tau\omega)\right) c_V^{\mathfrak{a}}\left(\eta^{-2}\omega\right) \quad \text{ for } \omega \in \mathfrak{O} - \{0\}\,. \tag{1.7.16}$$

Remark 1.7.1 (on spectral parameters). Except in those cases where the elements of V are functions with character $\Upsilon_{0,0}$ (so that, by (1.7.10) and (1.3.3), one has $\nu_V = p_V = 0$), the procedure for assigning spectral parameters to V indicated in (1.7.6) will yield exactly two choices for the spectral parameters (ν_V, p_V) of V, with the two choices in question, (ν', p') and (ν'', p'') (say), satisfying the relation $\nu' + \nu'' = p' + p'' = 0$. Consequently one may substitute $(-\nu_V, -p_V)$ for (ν_V, p_V) in all of the points covered between (1.7.7) and (1.7.16) (always provided that the appropriate operator on $H(-\nu_V, -p_V)$ is substituted for T_V).

We assume henceforth that each irreducible subspace $V \subset {}^0L^2(\Gamma\backslash G)$ is assigned a specific choice of spectral parameters (ν_V, p_V). However, since the essential points of what follows in this paper are independent of the choices made in the course of assigning those spectral parameters, we allow that those choices may be made arbitrarily. With regard to this note that, by (1.7.8), (1.6.5), (1.6.6), (1.7.13)-(1.7.15) and the functional equation

$$(\pi|\omega|)^{-\nu}\,(i\omega/|\omega|)^p\,\Gamma(\ell+1+\nu)\mathbf{J}_\omega\varphi_{\ell,q}(\nu,p) = (\pi|\omega|)^{\nu}\,(i\omega/|\omega|)^{-p}\,\Gamma(\ell+1-\nu)\mathbf{J}_\omega\varphi_{\ell,q}(-\nu,-p) \qquad (1.7.17)$$

(which is Equation (5.29) of [5]), it follows that the operator $T_V : H(\nu_V, p_V) \to V_K$ and its counterpart with domain $H(-\nu_V, -p_V)$ determine a real constant $\phi \in [0, 2\pi)$ such that, for all non-zero $\omega \in \mathfrak{O}$ and all cusps $\mathfrak{c}$ of Γ, one has $C_V^{\mathfrak{c}}(\omega; \nu_V, p_V) = e^{i\phi}C_V^{\mathfrak{c}}(\omega; -\nu_V, -p_V)$. Therefore it is in particular the case that each summand of the sum over V occurring in the Spectral Sum Formula of Theorem B (below) is unchanged if $(-\nu_V, -p_V)$ is substituted for (ν_V, p_V) (it being assumed here that the relevant function h satisfies the condition (i) of Theorem B); the same is true of the summands of that sum over V which occurs in the definition of $E_0^{\mathfrak{a}}(q_0, P, K; N, b)$ given in Theorem 1 (below).

§1.8 Decomposing the subspace ${}^{\mathrm{e}}L^2(\Gamma\backslash G)$: the Eisenstein series and a Parseval identity.

The subspace ${}^{\mathrm{e}}L^2(\Gamma\backslash G)$ in (1.7.3) is generated by integrals of certain Eisenstein series. To obtain a set of these series, sufficient for the generation of ${}^{\mathrm{e}}L^2(\Gamma\backslash G)$, first choose (once and for all) a complete set of representatives $\mathfrak{C}(\Gamma)$ of the Γ-equivalence classes of cusps, and, for each $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, a scaling matrix $g_{\mathfrak{c}}$ satisfying (1.1.16)-(1.1.21). Then, for $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, $\ell, p, q \in \mathbb{Z}$ with $\ell \geq |p|, |q|$ and $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > 1$, the Eisenstein series $E_{\ell,q}^{\mathfrak{c}}(\nu,p) : G \to \mathbb{C}$ is given by:

$$E_{\ell,q}^{\mathfrak{c}}(\nu,p)(g) = \frac{1}{[\Gamma_{\mathfrak{c}} : \Gamma_{\mathfrak{c}}']} \sum_{\gamma \in \Gamma_{\mathfrak{c}}' \backslash \Gamma} \varphi_{\ell,q}(\nu,p)\left(g_{\mathfrak{c}}^{-1}\gamma g\right) \quad \text{for } g \in G\,, \qquad (1.8.1)$$

where $\varphi_{\ell,q}(\nu,p) \in C^\infty(N\backslash G)$ is as in (1.3.2). By (1.1.20)-(1.1.21), the sum in (1.8.1) is well-defined; the results on the K-trivial case $\ell = p = q = 0$ in [11], Proposition 3.1.3, Proposition 3.2.1, Proposition 3.2.3 and Corollary 3.1.6, imply that, while this sum is divergent (for almost all $g \in G$) when $\nu \leq 1$, it does converge uniformly (and absolutely) for the pairs $(\nu, g) \in \mathbb{C} \times Na[r]K$ with $\mathrm{Re}(\nu) \geq 1 + \varepsilon$ and $r \geq \varepsilon$, where ε is an arbitrary positive constant. The definition (1.8.1) ensures that $E_{\ell,q}^{\mathfrak{c}}(\nu,p)$ is a Γ-automorphic function: it moreover inherits from $\varphi_{\ell,q}(\nu,p)$ the property of being a function of K-type (ℓ, q) with character $\Upsilon_{\nu,p}$.

By (1.3.2) and (1.3.4), one has

$$\varphi_{\ell,q}(\nu,p)\,(nh[u]g) = |u|^{2(1+\nu)}\,(u/|u|)^{-2p}\,\varphi_{\ell,q}(\nu,p)(g) \quad \text{for } u \in \mathbb{C} - \{0\},\, n \in N\,. \qquad (1.8.2)$$

Consequently, for cusps $\mathfrak{c}$ with $g_{\mathfrak{c}}^{-1}\Gamma_{\mathfrak{c}}g_{\mathfrak{c}} \cap h[i]N \neq \emptyset$, the sum in (1.8.1) will equal zero whenever p is odd. Since $[\Gamma_{\mathfrak{c}} : \Gamma_{\mathfrak{c}}'] = 4$ if $g_{\mathfrak{c}}^{-1}\Gamma_{\mathfrak{c}}g_{\mathfrak{c}} \cap h[i]N \neq \emptyset$, while $[\Gamma_{\mathfrak{c}} : \Gamma_{\mathfrak{c}}'] = 2$ otherwise, it therefore follows that

$$E_{\ell,q}^{\mathfrak{c}}(\nu,p) \neq 0 \quad \text{only if} \quad p \in \tfrac{1}{2}\,[\Gamma_{\mathfrak{c}} : \Gamma_{\mathfrak{c}}']\,\mathbb{Z}\,. \qquad (1.8.3)$$

For $p \in \frac{1}{2}\,[\Gamma_{\mathfrak{c}} : \Gamma_{\mathfrak{c}}']\,\mathbb{Z}$ the definition (1.8.1) coincides with Definition 3.3.2 of [32].

It is almost immaterial exactly which representative $\mathfrak{c}$ and which scaling matrix $g_{\mathfrak{c}}$, are chosen (as above) for use in defining the Eisenstein series: for it follows by (1.8.1)-(1.8.3), Lemma 2.1 and Lemma 4.2 that a different choice of $\mathfrak{c}$ or $g_{\mathfrak{c}}$ (in respect of any one Γ-equivalence class of cusps) will merely replace $E_{\ell,q}^{\mathfrak{c}}(\nu,p)$ by a function equal to $\epsilon^p E_{\ell,q}^{\mathfrak{c}}(\nu,p)$, for some $\epsilon \in \mathfrak{O}^*$.

By (1.5.4), and since $\varphi_{\ell,q}(\nu,p) \in C^\infty(N\backslash G)$, the Eisenstein series of (1.8.1) is a Poincaré series, $(P^{\mathfrak{c}}\varphi_{\ell,q}(\nu,p))(g)$, to which one might hope the case $\omega = 0$ of (1.5.5)-(1.5.10) would apply; it can be shown to follow from the definition (1.3.2) that this hope is justified when one has $\mathrm{Re}(\nu) > 1$. Hence and by (1.4.1), (1.5.18), (1.8.2), (1.8.3), Lemma 4.2 (below) and the linearity inherent in the definition (1.5.2) one finds that, if $\mathfrak{a}, \mathfrak{b} \in \mathfrak{C}(\Gamma)$, and if $\ell, p, q \in \mathbb{Z}$ and $\nu \in \mathbb{C}$ are such that $\ell \geq \max\{|p|, |q|\}$, $\mathrm{Re}(\nu) > 1$ and $E_{\ell,q}^{\mathfrak{a}}(\nu,p) \neq 0$, then, for $g \in G$,

$$\begin{aligned} E_{\ell,q}^{\mathfrak{a}}(\nu,p)\,(g_{\mathfrak{b}}g) = \delta_{\mathfrak{a},\mathfrak{b}}^{\Gamma}\varphi_{\ell,q}(\nu,p)(g) + \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma_{\mathfrak{a}}']}\,D_{\mathfrak{a}}^{\mathfrak{b}}(0;\nu,p)\,\frac{\pi\Gamma(|p|+\nu)}{\Gamma(\ell+1+\nu)}\,\frac{\Gamma(\ell+1-\nu)}{\Gamma(|p|+1-\nu)}\,\varphi_{\ell,q}(-\nu,-p)(g) \,+ \\ + \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma_{\mathfrak{a}}']} \sum_{0 \neq \psi \in \mathfrak{O}} D_{\mathfrak{a}}^{\mathfrak{b}}\,(\psi;\nu,p)\,(\mathbf{J}_\psi\varphi_{\ell,q}(\nu,p))(g)\,, \qquad (1.8.4) \end{aligned}$$

where

$$\delta_{\mathfrak{a},\mathfrak{b}}^{\Gamma}=\begin{cases}1 & \text{if } \mathfrak{a} \overset{\Gamma}{\sim} \mathfrak{b},\\ 0 & \text{otherwise,}\end{cases} \tag{1.8.5}$$

and

$$D_{\mathfrak{a}}^{\mathfrak{b}}(\psi;\nu,p)=\sum_{c\in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} S_{\mathfrak{a},\mathfrak{b}}(0,\psi;c)\,|c|^{-2(1+\nu)}\,(c/|c|)^{2p}\ . \tag{1.8.6}$$

Note that Theorem 3.4.1 of [11], contains the K-trivial case of this Fourier expansion. The sums $S_{\mathfrak{a},\mathfrak{b}}(0,\psi;c)$ generalise the Ramanujan sum evaluated in [15], Theorem 271; one has in particular a better estimate for these sums than that provided by (1.5.12); it can consequently be shown that the sum in (1.8.6) is absolutely convergent when either $\psi=0$ and $\mathrm{Re}(\nu)>1$, or $0\neq\psi\in\mathfrak{O}$ and $\mathrm{Re}(\nu)>0$.

It follows by (1.3.2) and the uniform convergence of the series in (1.8.1) that when $\mathfrak{c}$, ℓ, p, q and g are given, the function $\nu\mapsto E_{\ell,q}^{\mathfrak{c}}(\nu,p)(g)$ is holomorphic for $\mathrm{Re}(\nu)>1$. Furthermore, it is known that this function of ν has a meromorphic continuation to all of $\mathbb{C}$ with (in the particular cases considered in this paper) a simple pole at $\nu=1$ if and only if $\ell=p=q=0$, and no other poles in the closed half plane $\{\nu\in\mathbb{C}:\mathrm{Re}(\nu)\geq 0\}$. Applying this meromorphic continuation for each $g\in G$, one obtains, when $\mathrm{Re}(\nu)\geq 0$ and $(\nu,p)\notin\{(0,0),(1,0)\}$, a function $E_{\ell,q}^{\mathfrak{c}}(\nu,p):G\rightarrow\mathbb{C}$ which lies in the space $C^{\infty}(\Gamma\backslash G)$ and inherits (from the functions $E_{\ell,q}^{\mathfrak{c}}(\nu',p)$ with $\mathrm{Re}(\nu')>1$) the properties of being of K-type (ℓ,q) with character $\Upsilon_{\nu,p}$. Due to the nature of the first two terms on the right-hand side of the Fourier expansion (1.8.4), the function $E_{\ell,q}^{\mathfrak{c}}(\nu,p)$ does not lie in the space $L^2(\Gamma\backslash G)$ (except, possibly, when $(\nu,p)=(0,0)$): this can be seen by evaluation of the integral $\int_{(z,r)\in\mathcal{E}_{\mathfrak{c}}}\int_{k\in K^{+}}|\varphi_{\ell,q}\left(g_{\mathfrak{c}}n[z]a[r]k\right)|^2\,r^{-3}\mathrm{d}_{+}z\,\mathrm{d}r\,\mathrm{d}k$, with $\mathcal{E}_{\mathfrak{c}}\subset\mathbb{H}_3$ as in (1.1.23). Therefore it is only by averaging $E_{\ell,q}^{\mathfrak{c}}(\nu,p)$ over a range of values of ν that one obtains an element of the space ${}^{e}L^2(\Gamma\backslash G)$ (see Theorem A below).

For trivial K-type (i.e. when $\ell=p=q=0$) the above facts concerning the meromorphic continuation of the Eisenstein series are established in [11], Theorem 6.1.2 and Theorem 6.1.11; this being achieved by means of an elegant general theory (valid when one substitutes for Γ any discrete subgroup $\Gamma'<SL(2,\mathbb{C})$ for which the corresponding findamental domain $\mathcal{F}'\subset\mathbb{H}_3$ is non-compact, yet of finite volume): the corresponding facts in respect of Eisenstein series of arbitrary K-type are contained in Langlands' even more general theory [31]. Unlike the general situation described in Proposition 6.2.2 of [11], there is here no possibility of a (finite) number of generators of the space ${}^{e}L^2(\Gamma\backslash G)$ arising from residues of the Eisenstein series $E_{0,0}^{\mathfrak{c}}(\nu,0)$ at a poles lying in the interval $(0,1]$, for the only pole of $E_{0,0}^{\mathfrak{c}}(\nu,0)$ with a positive real part is that at $\nu=1$, and the residue there is a function that is constant on G (and therefore orthogonal to ${}^{e}L^2(\Gamma\backslash G)$).

Alternative proof of the above remarks on meromorphic continuation of Eisenstein series may be obtained by detailed consideration of the particular coefficients $D_{\mathfrak{a}}^{\mathfrak{b}}(\psi;\nu,p)$ in (1.8.4) and (1.8.6). This requires an evaluation of the sum $S_{\mathfrak{a},\mathfrak{b}}(0,\psi;c)$ (analogous to the evaluation of the classical Ramanujan obtained in Theorem 271 of [15]), which turns out to be a not overly complicated affair in the case $\mathfrak{b}=\infty$. The result of this calculation (for $\mathfrak{b}=\infty$) leads, via (1.8.6), to an expression for $D_{\mathfrak{a}}^{\infty}(\psi;\nu,p)$ in terms of Hecke zeta-functions

$$\zeta\left(s,\lambda^{p/2}\chi\right)=\frac{1}{4}\sum_{0\neq\alpha\in\mathfrak{O}}\frac{\lambda^{p/2}(\alpha)\chi(\alpha)}{|\alpha|^{2s}}\qquad(\mathrm{Re}(s)>1), \tag{1.8.7}$$

where $\lambda^{m}(\alpha)=(\alpha/|\alpha|)^{4m}$ and either $s=\nu$ and $\chi:\mathfrak{O}\rightarrow\{1\}$, or $s=1+\nu$ and there is a primitive character $\tilde{\chi}:(\mathfrak{O}/d\mathfrak{O})^{*}\rightarrow\mathbb{C}^{*}$ such that $\chi(\alpha)=\tilde{\chi}(\mathcal{A})$ whenever $\alpha\in\mathcal{A}\in(\mathfrak{O}/d\mathfrak{O})^{*}$ (while $\chi(\alpha)=0$ if $|(\alpha,d)|>1$). It is therefore a corollary of Hecke's work in Section 6 of [17] that $D_{\mathfrak{a}}^{\infty}(\psi;\nu,p)$ can be meromorphically continued into all of $\mathbb{C}$; consequently one obtains, via (1.8.4) and Lemma 5.1 of [5], the meromorphic continuation of $E_{\ell,q}^{\mathfrak{a}}(\nu,g)(g)$. See Lemma 5.2 of [5] for an explicit determination of $D_{\infty}^{\infty}(\psi;\nu,p)$ in the case $q_0=1$ (i.e. for $\Gamma=SL(2,\mathfrak{O})$). By Lemma 5.1 of [5] (again), and by (1.4.1)-(1.4.2), (1.5.14), (1.5.15) and, for $\mathrm{Re}(\nu)>1$, the equation (1.8.4), the meromorphic continuation of $E_{\ell,q}^{\mathfrak{a}}(\nu,p)(g)$ implies that of $D_{\mathfrak{a}}^{\mathfrak{b}}(\psi;\nu,p)$ for all $\mathfrak{b}\in\mathfrak{C}(\Gamma)$ (i.e. not only in the special case $\mathfrak{b}=\infty$).

As the remarks of the last three paragraphs might suggest, the Eisenstein series enable one to describe a decomposition of the subspace ${}^{e}L^2(\Gamma\backslash G)$ in (1.7.3). By combining this decomposition with (1.7.4)-(1.7.8) (the decomposition of ${}^{0}L^2(\Gamma\backslash G)$) one can obtain a useful decomposition of $L^2(\Gamma\backslash G)$. One may work instead with restriction of the decomposition (1.7.3) to the subspace $L^2(\Gamma\backslash G;\ell,q)$ spanned by the elements in $L^2(\Gamma\backslash G)$ of K-type (ℓ,q). Then a key result is the following.

Theorem A (a Parseval identity). *Let $\ell, q \in \mathbb{Z}$ satisfy $\ell \geq |q|$, and suppose that $f_1, f_2 \in L^2(\Gamma\backslash G; \ell, q)$ are represented by bounded elements of $C^\infty(\Gamma\backslash G)$. Then, when $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, $j \in \{1,2\}$ and $p \in \mathbb{Z}$ with $|p| \leq \ell$, the inner product $\langle f_j, E^{\mathfrak{c}}_{\ell,q}(it,p)\rangle_{\Gamma\backslash G} = F^{\mathfrak{c}}_{j,p}(t)$ (say) is defined for all real t; the functions so defined, $F^{\mathfrak{c}}_{j,p} : \mathbb{R} \rightarrow \mathbb{C}$ ($\mathfrak{c} \in \mathfrak{C}(\Gamma)$, $j = 1,2$ and $p = -\ell, \ldots, \ell$), are each square-integrable with respect to the Lebesgue measure on $\mathbb{R}$, and one has:*

$$\begin{aligned}\langle f_1, f_2\rangle_{\Gamma\backslash G} = {} & \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \langle f_1, 1\rangle_{\Gamma\backslash G} \langle 1, f_2\rangle_{\Gamma\backslash G} + \\ & + \sum_{\substack{V \\ -\ell \leq p_V \leq \ell}} \frac{1}{\left\| T_V \varphi_{\ell,q}(\nu_V, p_V)\right\|^2_{\Gamma\backslash G}} \langle f_1, T_V \varphi_{\ell,q}(\nu_V, p_V)\rangle_{\Gamma\backslash G} \langle T_V \varphi_{\ell,q}(\nu_V, p_V), f_2\rangle_{\Gamma\backslash G} + \\ & + \sum_{\mathfrak{c} \in \mathfrak{C}(\Gamma)} \frac{[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]}{4\pi i} \sum_{\substack{p \in \frac{1}{2}[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]\mathbb{Z} \\ |p| \leq \ell}} \int_{(0)} \frac{1}{\left\|\varphi_{\ell,q}(\nu,p)\right\|^2_{\mathrm{ps}}} \left\langle f_1, E^{\mathfrak{c}}_{\ell,q}(\nu,p)\right\rangle_{\Gamma\backslash G} \left\langle E^{\mathfrak{c}}_{\ell,q}(\nu,p), f_2\right\rangle_{\Gamma\backslash G} \mathrm{d}\nu\,,\end{aligned} \tag{1.8.8}$$

where the '1' in both $\langle f_1, 1\rangle_{\Gamma\backslash G}$ and $\langle 1, f_2\rangle_{\Gamma\backslash G}$ denotes the constant function $\varphi_{0,0}(-1,0)$ defined by (1.3.2), while V, in the second summation on the right-hand side of the equation, runs over the pairwise-orthogonal cuspidal subspaces of $L^2(\Gamma\backslash G)$ that occur in the direct sum in (1.7.4), and the notation '(0)' below the integral sign signifies that the integration is along the line $\{\nu \in \mathbb{C} : \mathrm{Re}(\nu) = 0\}$, oriented as a contour from $-i\infty$ to $i\infty$. The sums and integrals in equation (1.8.8) are absolutely convergent.

Proof. These results are a special case of Theorem 8.1 of [32], and are (conversely) a slight generalisation of Theorem 8.1 of [5]. They are also a special case of the very general Parseval identity proved in [31] (see also [16]). As is noted in [32], the restriction of (1.8.8) to pairs of K-trivial functions f_1, f_2 is (effectively) a result contained in Theorem 6.3.4 of [11] ■

The definition (1.8.6) of the Fourier coefficients $D^{\mathfrak{b}}_{\mathfrak{a}}(\omega; \nu, p)$ is not only valid for cusps in the particular set of representatives $\mathfrak{C}(\Gamma)$: it is in fact the appropriate definition for a arbitrary pair of cusps $\mathfrak{a}, \mathfrak{b}$ of Γ. For if $\mathfrak{a}$ and $\mathfrak{b}$ are not Γ-equivalent then one may assume that $\mathfrak{C}(\Gamma) \supseteq \{\mathfrak{a}, \mathfrak{b}\}$; while if instead $\mathfrak{a} \overset{\Gamma}{\sim} \mathfrak{b}$, then one may assume that $\mathfrak{C}(\Gamma) \ni \mathfrak{b}$, and so make use of the fact that $E^{\mathfrak{a}}_{\ell,q}(\nu,p)\,(g_{\mathfrak{b}} g) = \epsilon^p E^{\mathfrak{b}}_{\ell,q}(\nu,p)\,(g_{\mathfrak{b}} g)$ for some $\epsilon \in \mathfrak{O}^*$. Indeed, apart from the coefficient $\delta^{\Gamma}_{\mathfrak{a},\mathfrak{b}}$ in (1.8.4) possibly being replaced by $\epsilon^p \delta^{\Gamma}_{\mathfrak{a},\mathfrak{b}}$, for some $\epsilon \in \mathfrak{O}^*$, the whole Fourier expansion (1.8.4)-(1.8.6) is valid for arbitrary cusps $\mathfrak{a}, \mathfrak{b}$: and this does not require that $\mathfrak{a} = \mathfrak{b}$ imply $g_{\mathfrak{a}} = g_{\mathfrak{b}}$. A useful normalisation of $D^{\mathfrak{b}}_{\mathfrak{a}}(\psi; \nu, p)$ (analogous to (1.7.15)) is given by:

$$B^{\mathfrak{b}}_{\mathfrak{a}}(\omega; \nu, p) = (\pi|\omega|)^{\nu} \left(\omega/|\omega|\right)^{-p} D^{\mathfrak{b}}_{\mathfrak{a}}(\omega; \nu, p)\,. \tag{1.8.9}$$

§1.9 Results and applications.

The principal new results of this paper depend on being able to deduce estimates for a certain mean-value of Fourier coefficients of Γ-automorphic cusp forms and Eisenstein series from suitable bounds for sums of the generalised Kloosterman sums defined in (1.5.10). This is made possible by the following result.

Theorem B (spectral sum formula). *Let the real numbers $\sigma \in (1/2, 1)$, $\varrho, \vartheta \in (3, \infty)$, and the function $h : \{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{Z} \rightarrow \mathbb{C}$, satisfy the three conditions*

- *(i)* $h(\nu, p) = h(-\nu, -p)$;
- *(ii) for $p \in \mathbb{Z}$, the function $\nu \mapsto h(\nu, p)$ can be holomorphically continued into a neighbourhood of the strip $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\}$;*
- *(iii)* $h(\nu, p) \ll_{h,\varrho,\vartheta} (1 + |\mathrm{Im}(\nu)|)^{-\varrho}(1 + |p|)^{-\vartheta}$.

Suppose moreover that $0 \neq q_0 \in \mathfrak{O} = \mathbb{Z}[i]$, and that $\Gamma = \Gamma_0(q_0) \leq SL(2, \mathfrak{O})$. Then, for all $\omega_1, \omega_2 \in \mathfrak{O} - \{0\}$, all pairs of cusps $\mathfrak{a}, \mathfrak{b}$ of Γ, and all choices of the associated scaling matrices $g_{\mathfrak{a}}, g_{\mathfrak{b}}$ that satisfy the conditions

(1.1.16) and (1.1.20)-(1.1.21), one has

$$\sum_V \overline{C_V^{\mathfrak{a}}\left(\omega_1;\nu_V,p_V\right)}\, C_V^{\mathfrak{b}}\left(\omega_2;\nu_V,p_V\right) h\left(\nu_V,p_V\right) \ + \\ + \sum_{\mathfrak{c}\in\mathfrak{C}(\Gamma)} \frac{1}{4\pi i\left[\Gamma_{\mathfrak{c}}:\Gamma_{\mathfrak{c}}'\right]} \sum_{p\in\frac{1}{2}[\Gamma_{\mathfrak{c}}:\Gamma_{\mathfrak{c}}']\mathbb{Z}} \int_{(0)} \overline{B_{\mathfrak{c}}^{\mathfrak{a}}\left(\omega_1;\nu,p\right)}\, B_{\mathfrak{c}}^{\mathfrak{b}}\left(\omega_2;\nu,p\right) h(\nu,p)\, \mathrm{d}\nu = \\ = \frac{1}{4\pi^3 i}\,\delta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}} \sum_{p\in\mathbb{Z}} \int_{(0)} h(\nu,p)\left(p^2-\nu^2\right)\,\mathrm{d}\nu \ + \\ + \sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}\left(\omega_1,\omega_2;c\right)}{|c|^2}\,(\mathbf{B}h)\!\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c}\right), \tag{1.9.1}$$

where

$$\delta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}} = \sum_{\substack{\gamma\in\Gamma_{\mathfrak{a}}'\backslash\Gamma\,:\,\gamma\mathfrak{b}=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}=\begin{pmatrix} u(\gamma) & \beta(\gamma) \\ 0 & 1/u(\gamma)\end{pmatrix}}} \mathrm{e}\left(\mathrm{Re}\left(\beta(\gamma)u(\gamma)\omega_1\right)\right)\delta_{u(\gamma)\omega_1,\omega_2/u(\gamma)}\ , \tag{1.9.2}$$

other notation is as developed in (1.1.17)-(1.1.19), (1.5.6), (1.5.8)-(1.5.10), (1.7.4), (1.7.13)-(1.7.15), (1.8.1), (1.8.6) and (1.8.9), and

$$(\mathbf{B}h)(z) = \frac{1}{4\pi i}\sum_{p\in\mathbb{Z}}\int_{(0)} \mathcal{K}_{\nu,p}(z)h(\nu,p)\left(p^2-\nu^2\right)\,\mathrm{d}\nu\ , \tag{1.9.3}$$

with

$$\mathcal{K}_{\nu,p}(z) = \frac{1}{\sin(\pi\nu)}\ \left(\mathcal{J}_{-\nu,-p}(z)-\mathcal{J}_{\nu,p}(z)\right)\ , \tag{1.9.4}$$

$$\mathcal{J}_{\nu,p}(z) = |z/2|^{2\nu}\left(z/|z|\right)^{-2p} J_{\nu-p}^*(z)J_{\nu+p}^*\left(\overline{z}\right)\ , \tag{1.9.5}$$

and

$$J_\xi^*(z) = \sum_{m=0}^{\infty}\frac{(-1)^m\,(z/2)^{2m}}{\Gamma(m+1)\Gamma(\xi+m+1)}\ . \tag{1.9.6}$$

In (1.9.1) the set of representatives $\mathfrak{C}(\Gamma)$ of the Γ-equivalence classes of cusps may be chosen independently of the given pair of cusps $\mathfrak{a},\mathfrak{b}$; and nothing more than (1.1.16) and (1.1.20)-(1.1.21) need be assumed in respect of the choice of scaling matrices $g_{\mathfrak{c}}$ for $\mathfrak{c}\in\mathfrak{C}(\Gamma)$ (even in the event that $\mathfrak{C}(\Gamma)\cap\{\mathfrak{a},\mathfrak{b}\}\neq\emptyset$): similarly, $g_{\mathfrak{a}}$ is allowed to differ from $g_{\mathfrak{b}}$, even when $\mathfrak{a}=\mathfrak{b}$. All sums and integrals occurring in the equations (1.9.1) and (1.9.3) are absolutely convergent; the sum occurring in Equation (1.9.2) has at most finitely many terms.

Remark 1.9.1 (on the proof of Theorem B). Theorem B is an extension of Bruggeman and Motohashi's Spectral-Kloosterman sum formula, Theorem 10.1 of [5], which pertains to the case in which one has $\Gamma = SL(2,\mathbb{Z}[i])$ (so that there is only one Γ-equivalence class of cusps). It builds also upon the work of Lokvenec-Guleska who, in Theorem 11.3.3 of [32], succeeded in generalising Bruggeman and Motohashi's method so as to obtain a Spectral Kloosterman sum formula for Hecke congruence subgroups over an arbitrary imaginary quadratic field. Theorem 11.3.3 of [32] contains the case $\mathfrak{a}=\mathfrak{b}=\infty$ of Theorem B. A proof of Theorem B is described in an appendix to this paper; in this proof the relevant steps of [4] and [32] are adapted so as to deal with any choice of the cusps $\mathfrak{a},\mathfrak{b}$.

Remark 1.9.2 (on a result of Kim and Shahidi). Given the points noted in Subsection 1.7 (see, in particular, the case $p_V=\ell=q=0$ of (1.7.10), and what is discussed between (1.7.5) and (1.7.8)), it follows from the result (1.4.15) of Kim and Shahidi that in the first summation in Equation (1.9.1) the spectral parameters ν_V,p_V of the relevant subspaces $V\subset{}^0L^2(\Gamma\backslash G)$ must, in each instance, satisfy

$$\text{either}\quad (\nu_V,p_V)\in(i\mathbb{R})\times\mathbb{Z}\ ,\quad\text{or else}\quad p_V=0\ \text{ and }\ \nu_V\in[-2/9,2/9]\ . \tag{1.9.7}$$

Remark 1.9.3 (on a Bessel function). By assigning a fixed value to either one of the variables in (1.9.6), one obtains a single variable complex function (i.e. either $z \mapsto J_\xi^*(z)$ or $\xi \mapsto J_\xi^*(z)$) that is holomorphic on $\mathbb{C}$. When ξ is not an integer, two linearly independent solutions of Bessel's differential equation, $x^2y'' + xy' + (x^2 - \xi^2)y = 0$ $(x > 0)$, are $y_1 = J_\xi(x)$ and $y_2 = J_{-\xi}(x)$, where

$$J_\nu(z) = (z/2)^\nu J_\nu^*(z) \tag{1.9.8}$$

(this function $J_\nu(z)$ being Bessel's function of order ν). When ξ is an integer, the equations $y_1 = J_\xi(x)$ and $y_2 = J_{-\xi}(x)$ (with $J_\nu(z)$ as in (1.9.8)) do define solutions of Bessel's differential equation, but these solutions are linearly dependent, for it follows from (1.9.8) and (1.9.6) that

$$J_{-n}(z) = (-1)^n J_n(z) = J_n(-z) \qquad (n \in \mathbb{Z},\ z \in \mathbb{C}). \tag{1.9.9}$$

Remark 1.9.4 (on a partial inversion of the operator B). Let the function $f : \mathbb{C}^* \to \mathbb{C}$ be compactly supported and even. Suppose moreover that f is 'smooth', in the sense that every partial derivative (of whatever order) of the function $(x, y) \mapsto f(x + iy)$ is defined and continuous on the set $\mathbb{R}^2 - \{(0, 0)\}$. Then, as is shown in Theorem 11.1 of [5], one has:

$$\pi \mathbf{B}\mathbf{K} f = f \tag{1.9.10}$$

where the operator $\mathbf{B}$ is that given by (1.9.3)-(1.9.6), while

$$(\mathbf{K}f)(\nu, p) = \int_{\mathbb{C}^*} \mathcal{K}_{\nu,p}(z) f(z) |z|^{-2} \mathrm{d}_+ z \quad \text{for } p \in \mathbb{Z} \text{ and } \nu \in \mathbb{C} \text{ with } |\mathrm{Re}(\nu)| < 1\,.$$

By applying Theorem B with $h = \mathbf{K}f$ one obtains (using (1.9.10)) the corollary that the sum

$$L_{\mathfrak{a},\mathfrak{b}}(\omega_1, \omega_2; f) = \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1, \omega_2; c)}{|c|^2} f\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c}\right)$$

may be expressed in terms of sums involving Fourier coefficients of Γ-automorphic cusp forms and Eisenstein series. By Lemma 11.1 of [5], this inversion of the summation formula (1.9.1) contains no 'diagonal term' (i.e. no counterpart of the term in (1.9.1) with coefficient $\delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}$); it is in fact only a one-sided (non-surjective) inversion, since, as is noted in Section 11 of [5], there are test functions h satisfying the conditions (i)-(iii) of Theorem B that do produce a non-zero diagonal term on the right-hand side of (1.9.1). This inversion of Theorem B is not needed in this paper, but is important in [46] and [47].

Remark 1.9.5. Bruggeman and Motohashi showed in Theorem 12.1 of [5] that when $0 \neq z \in \mathbb{C}$, when $e^{i\theta} = z/|z|$ (so that $\theta \in \mathbb{R}$), and when one defines $\mathcal{K}_{\nu,p}(z)$ by (1.9.4)-(1.9.6), it then follows that

$$\mathcal{K}_{\nu,p}(u) = \frac{(-1)^p}{\pi/2} \int_0^\infty y^{2\nu} \left(\frac{ye^{i\theta} + (ye^{i\theta})^{-1}}{|ye^{i\theta} + (ye^{i\theta})^{-1}|}\right)^{2p} J_{2p}\bigl(|u|\,\bigl|ye^{i\theta} + (ye^{i\theta})^{-1}\bigr|\bigr) \frac{\mathrm{d}y}{y} \qquad (p \in \mathbb{Z},\ |\mathrm{Re}(\nu)| < \tfrac{1}{4}). \tag{1.9.11}$$

In proving Theorem 1 one needs to consider, for a suitable test function h, the transform $\mathbf{B}h$ that is defined in (1.9.3). Useful approximations to the relevant transformed function $(\mathbf{B}h)(z)$ may be deduced with the aid of both the identity (1.9.11) and an addition law for Bessel functions (see Lemma 4.5 and the proof of Lemma 4.6, below).

The principal new result in this paper is Theorem 1, stated next. We prove this theorem in Section 5 of this paper, with the help of certain bounds for sums of Kloosterman sums. These bounds are supplied by Proposition 2 (which we state after several remarks following Theorem 1).

Theorem 1. *Let $\varepsilon > 0$, $0 \neq q_0 \in \mathfrak{O} = \mathbb{Z}[i]$, $\Gamma = \Gamma_0(q_0) \leq SL(2,\mathfrak{O})$ and $K, P, N \geq 1$. Suppose further that $b : \mathfrak{O} - \{0\} \rightarrow \mathbb{C}$, and that $u, w \in \mathfrak{O}$ satisfy $w \neq 0$ and $(u,w) \sim 1$. Then, when $\mathfrak{a}$ is a cusp of Γ with $\mathfrak{a} \stackrel{\Gamma}{\sim} u/w$, and when $E_0^{\mathfrak{a}}(q_0, P, K; N, b)$, $E_1^{\mathfrak{a}}(q_0, P, K; N, b)$ are the quadratic moments given by*

$$E_0^{\mathfrak{a}}(q_0, P, K; N, b) = \sum_{\substack{V \\ |p_V| \leq P,\ |\nu_V| \leq K}} \Bigg| \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} b(\omega) C_V^{\mathfrak{a}}(\omega; \nu_V, p_V) \Bigg|^2 , \tag{1.9.12}$$

$$E_1^{\mathfrak{a}}(q_0, P, K; N, b) = \sum_{\mathfrak{c} \in \mathfrak{C}(\Gamma)} \frac{1}{4\pi \left[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}\right]} \sum_{\substack{p \in \frac{1}{2}\left[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}\right]\mathbb{Z} \\ |p| \leq P}} \int_{-K}^{K} \Bigg| \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} b(\omega) B_{\mathfrak{c}}^{\mathfrak{a}}(\omega; it, p) \Bigg|^2 \mathrm{d}t \tag{1.9.13}$$

(where the terminology used has the same meaning as in Theorem B), one has the upper bounds:

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \ll \left(P^2 + K^2\right) \left(PK + O_\varepsilon\left(\frac{N^{1+\varepsilon}}{(PK)^{1/2}} |\mu(\mathfrak{a})|^2 \right) \right) \|\mathbf{b}_N\|_2^2 \qquad (j = 0, 1)\,, \tag{1.9.14}$$

where $\mu(\mathfrak{a}) \in \{1/\alpha : 0 \neq \alpha \in \mathfrak{O}\}$,

$$\frac{1}{\mu(\mathfrak{a})} \sim \frac{(w, q_0)\, q_0}{(w^2, q_0)} \sim \frac{q_0}{\left((w, q_0)\,, q_0/(w, q_0)\right)} \tag{1.9.15}$$

and

$$\|\mathbf{b}_N\|_2 = \Bigg(\sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} |b(\omega)|^2 \Bigg)^{1/2} . \tag{1.9.16}$$

Remark 1.9.6. Since $[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}] \in \{2, 4\}$ for all cusps $\mathfrak{c}$, the factor $(4\pi\,[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}])^{-1}$ in (1.9.13) may be omitted.

Remark 1.9.7. One may check that (1.9.15) makes the ideal $(1/\mu(\mathfrak{a}))\mathfrak{O}$ a function of the Γ-equivalence class of the cusp $\mathfrak{a}$. The same is therefore true of the reciprocal of the norm of this ideal, which is the factor $|\mu(\mathfrak{a})|^2$ appearing in (1.9.14). Since $\infty \stackrel{\Gamma}{\sim} 1/q_0$ (for $\Gamma = \Gamma_0(q_0)$), one has in particular $1/\mu(\infty) \sim 1/\mu(1/q_0) \sim q_0$.

Remark 1.9.8 (some comparisons and conjectures). In their proof of Corollary 10.1 of [5], Bruggeman and Motohashi show, by an application of their 'Spectral-Kloosterman' sum formula (Theorem 10.1 of [5]), that if $\Gamma = SL(2,\mathbb{Z}[i])$ (so that $q_0 \sim 1$) then, for $0 \neq \omega \in \mathfrak{O}$, $P \geq 1$, $K \geq 1$ and $\varepsilon > 0$, one has

$$\begin{aligned} \sum_V \left| C_V^{\infty}(\omega : \nu_V, p_V)\right|^2 \exp\!\left((\nu/K)^2 - (p/P)^2\right) = \frac{1}{4\pi^2}\, \left(P^2 + K^2\right) PK \left(1 + O\left(P^2 e^{-\pi^2 P^2}\right)\right) \ + \\ + O_\varepsilon\left(|\omega|^{1+\varepsilon} \left(P^2 + K^2\right)^{1+\varepsilon} \right) . \end{aligned} \tag{1.9.17}$$

We may compare this with the result (5.55) obtained in the course of our proof of Theorem 1. Our result (5.55) is certainly weaker than (1.9.17) in cases where $|\omega|^2 > (PK)^{1+\varepsilon}$. However it follows from (5.55) that when $PK > |\omega|^2$ one may substitute $O_\varepsilon(|\omega|^2 (PK)^{-1/2} (P^2 + K^2)^{1+\varepsilon})$ for the final O_ε-term in Equation (1.9.17). The proof of this does require certain estimates for the relevant instances of the sum occurring on the final line of Equation (5.3); estimates sufficient for this purpose were already obtained in the proof of Corollary 10.1 of [5], by means of the lower bounds $|\zeta(1 + \nu, \lambda^{p/2})| \gg 1/\log(|\nu| + |p| + 2)$ $(p \in 2\mathbb{Z},\ \nu \in i\mathbb{R})$.

Based on considerations relating to the Eisenstein series $E_{0,0}^{\mathfrak{a}}(\nu, 0)$ and Rankin-Selberg function,

$$L_V^{\mathfrak{a}}(s) = \sum_{0 \neq \omega \in \mathfrak{O}} \left| c_V^{\mathfrak{a}}(\omega)\right|^2 |\omega|^{-2s} ,$$

we conjecture that in general (i.e. not just for $q_0 \sim 1$) it is the case that if, as in Lemma 2.2 (below), one has $\mathfrak{a} \stackrel{\Gamma}{\sim} u/w$ for some $u, w \in \mathfrak{O}$ satisfying $(u,w) \sim 1$, $u \neq 0$ and $w \mid q_0$, then

$$\sum_{\substack{\omega \in \mathfrak{O} \\ 0<|\omega|^2 \leq N}} \left|C_V^{\mathfrak{a}}(\omega : \nu_V, p_V)\right|^2 \sim \rho(\Gamma, \mathfrak{a}) N \qquad \text{as} \quad N \to \infty\ , \tag{1.9.18}$$

where

$$\rho(\Gamma, \mathfrak{a}) = \frac{2}{\operatorname{vol}(\Gamma \backslash G)} \prod_{\substack{\varpi \in (\mathfrak{O} - \{0\})/\mathfrak{O}^* \\ \varpi \mathfrak{O} \text{ is prime} \\ \varpi \mid (q/w, w/(q/w,w))}} \left(1 - |\varpi|^{-2}\right)$$

(so that one has, in particular, $\rho(\Gamma, \infty) = 2/\operatorname{vol}(\Gamma \backslash G)$). We believe that this conjecture might be shown to be correct by methods analogous to those described in Section 8.2 of [22].

Returning from the conjectural to the proven we note that, through an application of the Spectral-Kloosterman summation formula obtained in Theorem 11.3.3 of [32], Lokvenec-Guleska obtains, in Section 11.5 of [32], new asymptotic estimates for sums involving modified Fourier coefficients of cusp forms. In this work of Lokvenec-Guleska '$\mathfrak{O}$' denotes the ring of integers of an arbitrary quadratic number field $F = \mathbb{Q}(\sqrt{-d})$ (with $d \in \mathbb{N}$), and 'Γ' denotes a Hecke congruence subgroup of $SL(2, \mathfrak{O})$ associated with some (arbitrary) non-zero ideal $I \subseteq \mathfrak{O}$; her definition of automorphicity is broader than ours, in that it depends on a character $\chi : \Gamma \to \mathbb{C}$ derived from an arbitrary character for $(\mathfrak{O}/I)^*$; the relevant irreducible cuspidal subspaces (corresponding to the spaces denoted by 'V' in our paper) contain automorphic functions $f : G \to \mathbb{C}$ that are even if $\chi(-1) = 1$, but odd if $\chi(-1) = -1$. We state here only one of Lokvenec-Guleska's results, specialised to the particular case $\mathfrak{O} = \mathbb{Z}(i)$, $I = q_0\mathfrak{O}$, $\chi : \Gamma \to \{1\}$, in which her '$L^2(\Gamma \backslash G, \chi)$' coincides with the space $L^2(\Gamma \backslash G)$ that we have defined. In respect of this case, it follows by the combination of Part (i) of Theorem 11.5.2 of [32] and the symmetry used in the deduction of Corollary 11.5.4 of [32] that, for $0 \neq \omega \in \mathfrak{O}$ and $p \in \mathbb{Z}$, one has

$$\sum_{\substack{V\ :\ p_V = p \\ 1 - \nu_V^2 - p_V^2 \leq X}} \left|C_V^{\infty}(\omega; \nu_V, p_V)\right|^2 \sim \frac{1}{3\pi^3} X^{3/2} \qquad \text{as} \quad X \to \infty\ . \tag{1.9.19}$$

The second condition of summation in (1.9.19) is motivated by the fact that $1 - \nu_V^2 - p_V^2$ is (by definition) an eigenvalue of the operator $-4(\Omega_+ + \Omega_-)$.

Theorem 11.5.2 of [32] is essentially a corollary (deduced via a Tauberian argument) of the asymptotic estimates for smoothly weighted sums which Lokvenec-Guleska obtains in Proposition 11.5.1 of [32]. Results very similar to the special case $p = 0$, $I = \mathfrak{O}$ ($\chi : \Gamma \to \{1\}$) of Proposition 11.5.1 of [32] had previously been obtained in the paper [39] of Rhagavan and Sengupta; the result (13) of [39] is a somewhat unwieldy special case of the spectral-Kloosterman sum formula obtained in Theorem 11.3.3 of [32].

The combination of the result (1.9.19) and our conjecture (1.9.18) are so suggestive as to lead us to make the further conjecture that, for $\Gamma = \Gamma_0(q_0) \leq SL(2, \mathbb{Z}[i])$ and $p \in \mathbb{Z}$, one has

$$\sum_{\substack{V\,:\,p_V = p \\ |\nu_V| \leq K}} 1 \sim \frac{\operatorname{vol}(\Gamma \backslash G)}{6\pi^2} K^3 \qquad \text{as} \quad K \to \infty\ . \tag{1.9.20}$$

This conjecture is at least partly correct: for the case $p = 0$ of (1.9.20) is a known instance in which Weyl's law holds (see [11], Page 308 and Section 8.9, for details of this). Moreover, what is hypothesised in (1.9.20) is equivalent to something that is (at least) superficially analogous to what Müller has proved in Theorem 0.1 of [37]. In fact, on the basis of the conjecture (1.9.18), and the form of the result which we obtain in (5.55) (below), we venture to put forward the conjecture that as $\delta \to 0+$ and $(p^2 + K^2)\delta K \to \infty$, with $p \in \mathbb{Z}$ and $\delta, K \in (0, \infty)$, one has

$$\sum_{\substack{V\,:\,p_V = p \\ K < |\nu_V| \leq K + \delta K}} 1 \sim \frac{\operatorname{vol}(\Gamma \backslash G)}{2\pi^2} \left(p^2 + K^2\right) \delta K \tag{1.9.21}$$

uniformly for all $\Gamma = \Gamma_0(q) \leq SL(2, \mathbb{Z}[i])$.

It is a significant feature of Theorem 1 that the bounds (1.9.14) hold for all cusps $\mathfrak{a}$ of Γ: a direct application of this feature occurs in the proof of Theorem 4 of [46], and indirect use of it is essential to the proof of Theorem 10 of [46]. However, in order to compare Theorem 1 with Lokvenec-Guleska's result (1.9.19), we now focus for a moment on the case $\mathfrak{a} = \infty$. If one were deprived of all information concerning the Fourier coefficients $C_V^{\infty}(\omega; \nu_V, p_V)$ other than what is stated in (1.9.19), then essentially the strongest upper bound for $E_0^{\infty}(q_0, P, K; N, b)$ that one could deduce would be the result

$$\limsup_{K \to \infty} K^{-3} E_0^{\infty}(q_0, P, K; N, b) \leq \frac{1}{6\pi^2} (2P+1)(N + o(N)) \, \|\mathbf{b}_N\|_2^2 \tag{1.9.22}$$

(in which $q_0 \in \mathfrak{O} - \{0\}$, $P, N \in \mathbb{N}$ and the function $b : \mathfrak{O} - \{0\} \to \mathbb{C}$ are presumed given). In fact the inequality (1.9.22) is simply what follows from (1.9.12) and (1.9.19) by means of the Cauchy-Schwarz inequality,

$$\Bigg| \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} b(\omega) C_V^{\mathfrak{a}}(\omega; \nu_V, p_V) \Bigg|^2 \leq \|\mathbf{b}_N\|_2^2 \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} \left| C_V^{\mathfrak{a}}(\omega; \nu_V, p_V) \right|^2 ,$$

and known estimates (either classical or more recent, as in [18]) for the number of lattice points lying in a disc of specified radius. The case $j = 0$ of (1.9.14) implies that the inequality (1.9.22) remains valid if, in place of the factor $N + o(N)$ on the right-hand side of that inequality, one substitutes just a factor $O(1)$; this reveals that the extent of 'cancellation' occurring within the sums over ω on the right-hand side of Equation (1.9.12) is, with few exceptions, comparable to that which one would expect to find, in the limit as $N \to \infty$, in respect of a sum

$$R(\mathcal{X}; N, b) = \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} b(\omega) \exp(i X_\omega)$$

in which $b(\omega)$ is a given complex-valued function, while $\mathcal{X} = (X_\omega)_{\omega \in \mathfrak{O} - \{0\}}$ is a family of independent real-valued random variables such that, for all $\omega \in \mathfrak{O} - \{0\}$, the mean value of $\exp(iX_\omega)$ is equal to zero.

Remark 1.9.9 (on improving on Theorem 1). The bounds in (1.9.14) are less precise than the asymptotic result (Equation (5.55), below), from which they are deduced. Moreover, in view of the positivity (exploited in (5.2) and (5.61), below) of the terms summed in (1.9.12) and (1.9.13), it is evident that Theorem 1 is not optimal when $PK = o\left(N^{2/3} |\mu(\mathfrak{a})|^{4/3}\right)$ and either $P = o(K)$ or $K = o(P)$. Indeed, by virtue of the positivity of those terms, it is a direct corollary of Theorem 1 itself that the result (1.9.14) may be improved to:

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \ll \Big(\left(P^2 + K^2\right) \Big(PK + O_\varepsilon\big(\mathcal{Y}^{2/3}\big) \Big) + O_\varepsilon\big((P+K)\mathcal{Y}\big) \Big) \|\mathbf{b}_N\|_2^2 \qquad (j = 0, 1) , \tag{1.9.23}$$

where $\mathcal{Y} = \mathcal{Y}(N, \varepsilon; q_0, \mathfrak{a}) = N^{1+\varepsilon} |\mu(\mathfrak{a})|^2$. On the basis of the conjecture (1.9.18), the conjecture (1.9.21) and the tendency towards cancellation observed in the final paragraph of Remark 1.9.8, we are led to make the further conjecture that Theorem 1 would remain true if the bound

$$E_0^{\mathfrak{a}}(q_0, P, K; N, b) \ll \left(\left(P^2 + K^2\right) PK + \frac{N}{\mathrm{vol}(\Gamma \backslash G)} \right) \|\mathbf{b}_N\|_2^2$$

were substituted in place of the result in the case $j = 0$ of (1.9.14). Therefore we believe that even the case $j = 0$ of (1.9.23) falls distinctly short of being best-possible. It also seems likely that Theorem 1 (or its corollary, (1.9.23)) could be improved upon in other ways: one might, for example, be able to prove a 'short-spectral-interval' refinement of the case $j = 0$ of (1.9.14) (i.e. a result analogous to the results obtained in Theorem 1.1 of [23], Theorem 3.3 of [35] and Lemma 7 of [24]); one might also be able to obtain, instead of just an upper bound, an asymptotic estimate for the sum $E_0^{\mathfrak{a}}(q_0, P, K; N, b)$.

We expect that the bound in the case $j = 1$ of the corollary (1.9.23) (and hence also the bound in the case $j = 1$ of (1.9.14)) can be significantly sharpened by exploiting the special nature of the modified Fourier coefficients $B_{\mathfrak{c}}^{\mathfrak{a}}(\omega; \nu, p)$ of the Eisenstein series: see Equation (10.1) of Theorem 10.1 of [5] for the simple explicit form that these coefficients take in the case $q_0 = 1$. The discussion around (1.8.7) casts further light on this interesting possibility. We have not ourselves attempted to improve upon our stated bounds for $E_1^{\mathfrak{a}}(q_0, P, K; N, b)$: it turns out that Theorem 1, as it stands, is adequate for our needs in [46] and [47].

Given what Lokvenec-Guleska achieved in her thesis [32], we are almost certain that methods similar to those of the present paper are capable of yielding, when $F \in \{\mathbb{Q}(-d) : d \in \mathbb{N},\ d > 1 \text{ and } d \text{ is squarefree}\}$ and Γ is any Hecke congruence subgroup of $SL(2, \mathfrak{O}_F)$ (where $\mathfrak{O}_F$ is equal to the ring of integers of F), both an analogue for $\Gamma\backslash SL(2,\mathbb{C})$ of the summation formula in Theorem B and results in some (useful) sense analogous to those in Theorem 1. Although such generalisations of Theorem B and Theorem 1 would be of considerable number-theoretical interest, we have (so far) preferred not to do any work in that direction ourselves: we found the case $F = \mathbb{Q}(\sqrt{-1})$ to be complicated enough.

Remark 1.9.10 (on related work of the author). This paper is the first in a planned series of three: the other two being [46] and [47]. Our work in [46] involves the application of Theorem 1, in combination with the result (1.4.15) of Kim and Shahidi and the the inversion of Theorem B mentioned in Remark 1.9.4; the results obtained there include new upper bounds for sums of the form

$$\sigma_{\Gamma}^{\mathfrak{a}}(\mathbf{b}, N; X) = \sum_{\substack{V \\ \nu_V^2 > 0,\ p_V = 0}} X^{|\nu_V|} \left| \sum_{\substack{n \in \mathfrak{O} \\ N/2 < |n|^2 \leq N}} b_n C_V^{\mathfrak{a}}(n; \nu_V, 0) \right|^2 \qquad (\mathfrak{a} \in \mathbb{Q}(i) \cup \{\infty\})$$

and

$$S(Q, X, N) = \sum_{\substack{q_1 \in \mathfrak{O} \\ Q/2 < |q_1|^2 \leq Q}} \sigma_{\Gamma_0(q_1)}^{\infty}(\mathbf{b}, N; X)\ ,$$

where X and N denote real numbers satisfying $X \geq 1$, $N \geq 1$, while the coefficients b_n $(0 \neq n \in \mathfrak{O})$ are complex numbers that are allowed to be arbitrary in [46], Theorem 4, Theorem 5, Theorem 6 and Theorem 7, but are required to be of a special type in [46], Theorem 8 and Theorem 9. Our results in [46] include analogues, for $SL(2,\mathbb{C})$, of several of the ($SL(2,\mathbb{R})$ related) results obtained by Deshouillers and Iwaniec in [9], and also the '$SL(2,\mathbb{C})$ analogue' of Theorem 2 of [45].

In work only partially written up we have made use of the results of [46] in estimating a particular type of weighted fourth power moment for the family of Hecke zeta functions $\left(\zeta(s, \lambda^k)\right)_{k \in \mathbb{Z}}$ given, for $\mathrm{Re}(s) > 1$, by the cases of (1.8.7) in which $\chi : \mathfrak{O} \to \{1\}$ and p is even. It was shown by Hecke, in [17], that these zeta functions have a meromorphic continuation to all of $\mathbb{C}$, with no poles except at $s = 1$ (and no pole there except when $k = 0$). By using the results of [46] we have been able to prove that, for arbitrary complex coefficients a_n $(0 \neq n \in \mathfrak{O})$, all $\varepsilon > 0$ and all $D, N \in \mathbb{N}$, one has

$$\sum_{k=-D}^{D} \int_{-D}^{D} \left|\zeta\left(\tfrac{1}{2} + it, \lambda^k\right)\right|^4 \left| \sum_{\substack{n \in \mathfrak{O} \\ 0 < |n|^2 \leq N}} a_n \lambda^k(n) |n|^{-2it} \right|^2 \mathrm{d}t \ll_{\varepsilon} D^{2+\varepsilon} N \max_{0 < |n|^2 \leq N} |a_n|^2 \qquad \text{if } N^2 \leq D.$$

We are preparing our proof of this result for publication, and hope that it may appear in [47].

Proposition 2. *Let ε, N, q_0 and b satisfy the same hypotheses as in Theorem 1, and let $\|\mathbf{b}_N\|_2$ be given by (1.9.16); let A_1 and A_2 be arbitrary positive absolute constants; let $\psi \in \mathbb{R}$ and $M \in \mathbb{N} \cup \{0\}$, and let $\mathfrak{a}$ be a cusp of $\Gamma = \Gamma_0(q_0) \leq SL(2, \mathfrak{O})$; let the 'scaling matrix' $g_{\mathfrak{a}} \in G = SL(2,\mathbb{C})$ be such as to satisfy, for $\mathfrak{c} = \mathfrak{a}$, each of (1.1.16), (1.1.20) and (1.1.21), and let c be an element of the set ${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}} \subset \mathbb{C}^*$ defined in (1.5.8), (1.5.9) (i.e. with $\mathfrak{a}' = \mathfrak{a}$ and $g_{\mathfrak{a}'} = g_{\mathfrak{a}}$ there). Then*

$$0 \neq c \in \frac{1}{\mu(\mathfrak{a})}\, \mathfrak{O} \tag{1.9.24}$$

(the relationship between $\mu(\mathfrak{a})$ and $\mathfrak{a}$ being as described in Remark 1.9.7, above), and the sum

$$U_{\mathfrak{a}}(\psi, c; M; N, b) = \sum_{m=-M}^{M} \left| \sum_{\substack{\omega_1,\omega_2 \in \mathfrak{O} \\ N/2 < |\omega_1|^2, |\omega_2|^2 \le N}} \overline{b\,(\omega_1)} b\,(\omega_2) \left(\frac{\omega_1 \omega_2}{|\omega_1 \omega_2|} \right)^m S_{\mathfrak{a},\mathfrak{a}}(\omega_1, \omega_2; c)\, \mathrm{e}\!\left(\frac{\psi \sqrt{|\omega_1 \omega_2|}}{|c|} \right) \right| \tag{1.9.25}$$

(with $S_{\mathfrak{a},\mathfrak{a}'}(\omega, \omega'; c)$ defined as in (1.5.8)-(1.5.10)) satisfies both

$$U_{\mathfrak{a}}(\psi, c; M; N, b) \ll \tau^{3/2}(c)|c|(M+1)N \left\| \mathbf{b}_N \right\|_2^2 \tag{1.9.26}$$

and

$$U_{\mathfrak{a}}(\psi, c; M; N, b) \ll (1 + |\psi|)^{1/2} \left(|c|\,(M+1) + N^{1/2} \right) \left(|c| + N^{1/2} \right) \left\| \mathbf{b}_N \right\|_2^2 \ , \tag{1.9.27}$$

where $\tau(c)$ is the number of Gaussian integer divisors of c.

If it is moreover the case that one has

$$0 < |c|^2 \le A_1 N^{1-\varepsilon} \tag{1.9.28}$$

and

$$0 < |\psi| \le A_2 \tag{1.9.29}$$

then

$$U_{\mathfrak{a}}(\psi, c; M; N, b) \ll_{A_1, A_2} \left(|\psi|^{-1/2} + O_{\varepsilon}(1) \right) \left(|c|^{1/2} N^{3/4} + |c|^{3/2} M N^{1/4} \right) N^{\varepsilon} \left\| \mathbf{b}_N \right\|_2^2 \ . \tag{1.9.30}$$

Proposition 2 is proved in Section 3: lemmas necessary for its proof are collected in Section 2.

Notation.

The following index of notation covers only some of the more unusual notation used in this paper; it is not comprehensive. Some of our other terminology is explained in five supplementary paragraphs.

Index of notation:

Symbol	**Description**	**Place defined**
$\lvert\mathcal{A}\rvert$	*(when $\mathcal{A}$ is a set): the cardinal number of $\mathcal{A}$*	–
$\lvert f\rvert$	*(when f is a complex valued function of x): the function $x \mapsto \lvert f(x)\rvert$*	–
R^*	*(when R is a ring with an identity): the group of units of R*	–
$\overline{U}$	*(when U is a subset of a metric or topological space): the closure of U*	–
$(g \circ f)(x)$	*equal to $g(f(x))$*	–
$a \cdot b$	*a multiplied by b ($(a \cdot b)(x) = a(x)b(x)$ if a and b are functions of x)*	–
$\mathbf{v} \cdot \mathbf{w}$	*equal to $v_1 w_1 + \cdots + v_n w_n$, the inner product of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$*	–
$\lvert z\rvert$ and $\overline{z}$	*the absolute value (or 'modulus') and complex conjugate of $z \in \mathbb{C}$*	–
$\mathfrak{a} \overset{\Gamma}{\sim} \mathfrak{b}$	*the relation of Γ-equivalence (for cusps $\mathfrak{a}, \mathfrak{b}$)*	above (1.1.16)
$m \mid n$	*(when $m, n \in \mathfrak{O}$): the relation 'n is divisible by m'*	–
$m \sim n$	*(when $m, n \in \mathfrak{O}$): the relation 'n is an associate of m'*	above (1.1.1)
$(m_1, \ldots, m_n)$	*a highest common factor (of $m_1, m_2, \ldots, m_n \in \mathfrak{O}$)*	–
(C, q_0^∞)	*a certain factor of the non-zero Gaussian integer C*	below (6.1.17)
$a \equiv b \bmod c\mathfrak{O}$	*equivalent to the statement that one has $a, b, c \in \mathfrak{O}$ and $c \mid (b-a)$*	–
$[x]$	*the greatest rational integer less than or equal to x*	–
$\lVert\beta\rVert$	*equal to* $\min\{\lvert n - \beta\rvert : n \in \mathbb{N}\}$	–
$\lVert\mathbf{b}_N\rVert_2$	*the Euclidean norm of a vector involving coefficients b_n ($0 \neq n \in \mathfrak{O}$)*	in (1.9.16)
$\int_{(\alpha)} f(z) \mathrm{d}z$	*contour integral along the straight line from $\alpha - i\infty$ to $\alpha + i\infty$*	–

$\langle f,h\rangle_{\Gamma\backslash G}$	*the inner product of* $f,g\in L^2(\Gamma\backslash G)$	in (1.2.2)
$\Vert f\Vert_{\Gamma\backslash G}$	*a norm on* $L^2(\Gamma\backslash G)$*, equal to* $\sqrt{\langle f,f\rangle_{\Gamma\backslash G}}$	–
$(f_1,f_2)_K$	*the inner product for square integrable functions* $f_1,f_2:K\rightarrow\mathbb{C}$	in (1.2.22)
$\Vert\Phi\Vert_K$	*a norm, equal to* $\sqrt{(\Phi,\Phi)_K}$	below (6.4.2)
$\langle f,F\rangle_{N\backslash G}$	*a certain inner product, defined when* $f\overline{F}\in L^1(N\backslash G)$	in (6.2.9)
$(f_1,f_2)_{\mathrm{ps}}$	*the inner product for the 'principal series'*	(1.6.2), (1.6.3)
$\Vert\varphi\Vert_{\mathrm{ps}}$	*a norm on the space* $H^2(\nu,p)$*, when* $(\nu,p)\in(i\mathbb{R})\times\mathbb{Z}$	below (1.6.4)
$(f_1,f_2)_{\mathrm{cs}}$	*the inner product for the 'complementary series'*	(1.6.2), (1.6.4)
$\Vert\varphi\Vert_{\mathrm{cs}}$	*a norm on the space* $H^2(\nu,p)$*, when* $0<\nu^2<1$	below (1.6.4)
$(f\vert\mathfrak{c})$	*a 'left-translate' of* $f:G\rightarrow\mathbb{C}$*, used for Fourier expansion of* f *at* $\mathfrak{c}$	above (1.4.2)
$\hat{F}(\mathbf{y})$, $\hat{f}(w)$	*Fourier transforms of* $F:\mathbb{R}^n\rightarrow\mathbb{C}$ *and* $f:\mathbb{C}\rightarrow\mathbb{C}$	(2.44), (2.46)
Γ, $\Gamma_0(q_0)$	*the Hecke congruence subgroup of* $SL(2,\mathfrak{O})$ *of 'level'* q_0	in (1.1.1)
$\Gamma_{\mathfrak{c}}$, $\Gamma'_{\mathfrak{c}}$	*'stabiliser' and 'parabolic stabiliser' subgroups (for the cusp* $\mathfrak{c}$*)*	(1.1.17)-(1.1.21)
${}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)$	*a 'Bruhat cell'*	in (1.5.8)
$\Gamma(z)$	*Euler's Gamma function, defined for* $z\in\mathbb{C}-\{0,-1,-2,\dots\}$	–
γ	*most often used to denote an element of the group* Γ	–
$\mathbf{\Delta}$	*the hyperbolic Laplacian operator on* $L^2(\Gamma\backslash\mathbb{H}_3)$	in (1.2.13)
$\mathbf{\Delta}_{\mathbb{R}\times\mathbb{R}}$, $\mathbf{\Delta}_{\mathbb{C}}$	*Euclidean Laplacian operators*	above (2.48)
$\partial/\partial z$, $\partial/\partial\overline{z}$	*Complex partial differentiation operators*	see (1.2.7)
$\delta^{\Gamma}_{\mathfrak{a},\mathfrak{b}}$	*the 'delta symbol' for* Γ*-equivalence of the cusps* $\mathfrak{a}$ *and* $\mathfrak{b}$	in (1.8.5)
$\delta^{\mathfrak{a},\mathfrak{b}}_{\omega,\omega'}$	*the 'delta symbol' of the 'spectral to Kloosterman' sum formula*	in (1.9.2)
$\delta_{w,z}$	*the 'delta-symbol' for equality of the complex numbers* w *and* z	in (1.5.6)
$\epsilon(p)$	*a certain function defined on* $\mathbb{Z}$*; takes values in* $\{-1,1\}$	in (6.4.5)
$\zeta(s)$	*the Riemann zeta-function*	–
$\zeta_{\mathbb{Q}(i)}(s)$	*the Dedekind zeta-function for the algebraic number field* $\mathbb{Q}(i)$	below (6.5.59)
$\zeta(s,\lambda^{p/2}\chi)$	*a Hecke zeta function (with grössencharakter) for* $\mathbb{Q}(i)$	in (1.8.7)
$\zeta^{\mathfrak{a},\mathfrak{b}}_{\omega,\omega'}(s)$	*a modified Linnik-Selberg series*	in (6.5.61)
$\kappa(\omega_1,\omega_2;c)$	*a linear operator from* $\mathcal{T}^{\ell}_{\sigma}$ *into* $\mathcal{T}^{\ell}_{\sigma}$	in (6.4.12)
$\lambda^{p/2}$, λ^k	*a Hecke grössencharakter on* $\mathfrak{O}-\{0\}$	below (1.8.7)
λ_{ν}	*an eigenvalue of the operator* $-\mathbf{\Delta}$ *on* $L^2(\Gamma\backslash\mathbb{H}_3)$	in (1.4.14)
$\lambda^*_{\ell}(\nu,p)$	*a certain function on the set* $\mathbb{C}\times\{p\in\mathbb{Z}:\vert p\vert\leq\ell\}$	in (6.6.4)
$\mu(\mathfrak{a})$	$1/\vert\mu(\mathfrak{a})\vert$ *is a useful lower bound for the set* $\{\vert c\vert:c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}\}$	in (1.9.15)
(ν_V,p_V)	*the spectral parameters of the cuspidal irreducible subspace* V	above (1.7.6)
$\rho:G\rightarrow(0,\infty)$	$\rho(g)$ *is equal to the Iwasawa coordinate* r *of the element* $g\in G$	(6.2.1), (1.1.2)
$\sum_V$	*summation over irreducible subspaces* $V\subset{}^0L^2(\Gamma\backslash G)$	below (1.8.8)
$\tau(n)$	*(when* $n\in\mathfrak{O}$*): the number of Gaussian integer divisors of* n	–
$\tau:G\rightarrow\mathbb{C}$	*an element of* $C^{\infty}(G)$ *chosen according to certain criteria*	see (6.5.1)
$\tau\mathbf{M}_{\omega}\varphi_{\ell,q}(\nu,0)$	*the product of the functions* τ *and* $\mathbf{M}_{\omega}\varphi_{\ell,q}(\nu,0)$	in (6.5.3)
$(1-\tau)\mathbf{M}_{\omega}\varphi$	*(when* $\varphi=\varphi_{\ell,q}(\nu,0)$*): the product of the functions* $1-\tau$ *and* $\mathbf{M}_{\omega}\varphi$	below (6.5.3)
$\Upsilon_{\nu,p}$	*a certain character for the centre of* $\mathfrak{g}$	in (1.3.3)
$\Phi^{\ell}_{p,q}$	*a certain even and square integrable function defined on* K	below (1.3.2)
$\varphi_{\ell,q}(\nu,p)$	*a certain function lying in the space* $C^{\infty}(N\backslash G)$	in (1.3.2)
$\phi(\alpha)$	*(when* $0\neq\alpha\in\mathfrak{O}$*): Euler's function, equal to* $\vert(\mathfrak{O}/\alpha\mathfrak{O})^*\vert$	below (2.5)
$\phi_{\omega'}(\nu,g)$	*an analytic continuation of a Fourier term,* $(F^{\mathfrak{b}}_{\omega'}P^{\mathfrak{a}}\mathbf{M}_{\omega}\varphi_{\ell,q}(\nu,0))(g)$	in (6.5.72)
$\Phi(\nu,g)$	*an analytic continuation of the function* $\nu\mapsto P^{\mathfrak{a}}\mathbf{M}_{\omega}\varphi_{\ell,q}(\nu,0))(g_{\mathfrak{b}}g)$	below (6.5.72)
ϕ_1, ϕ_2	*in §6.6 these denote certain pseudo Poincaré series*	in (6.6.1)
$\chi_4:\mathbb{N}\rightarrow\{-1,1\}$	*the real primitive Dirichlet character modulo* 4	–
$\chi_{q_0}:G\rightarrow\{0,1\}$	*the characteristic function of* $\Gamma_0(q_0)<G$	in (6.1.9)
$\chi^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$	*the 'delta-term' in the spectral sum formula*	in (6.7.1)
$\psi_{\omega}:N\rightarrow\mathbb{C}$	*a certain character for the group* N	in (1.4.3)
$\psi(y,x;\phi)$	*a certain elementary function*	in (4.23)

$\omega(c)$	*(when $0 \neq c \in \mathfrak{O}$): the number of prime ideals of $\mathfrak{O}$ containing c*	–
$\Omega_{\pm}$	*the Casimir operators associated with G*	in/below (1.2.7)
$\Omega_{\mathfrak{k}}$	*the Casimir operator associated with K*	in (1.2.11)
$\mathbf{x}, \mathbf{X}, \ \ldots$	*vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$; sets of coefficients; operators; power set*	–
$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$	*cusps of Γ, or (more generally) points in $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$*	above (1.1.16)
$a[r]$, A	$A = \{a[r] : r > 0\} < G$	see (1.1.3)
$A^0_{\Gamma}(\Upsilon; \ell, q)$	*the space of cusp forms in $C^{\infty}(\Gamma\backslash G)$ of K-type (ℓ, q) with character Υ*	in (1.4.6)
$A_M(\phi, \theta)$	*a certain trigonometric sum*	in (4.24)
$\mathrm{Arg}(z)$	*the principal argument of $z \in \mathbb{C}$, satisfying $-\pi < \mathrm{Arg}(z) \leq \pi$*	–
B^+	*the subgroup $\{n[\alpha] : \alpha \in \mathfrak{O}\} < N < G$*	in (1.1.21)
$B^{\mathfrak{b}}_{\mathfrak{a}}(\omega; \nu, p)$	*a modified Fourier coefficient of an Eisenstein series*	in (1.8.9)
$\mathbf{B}h : \mathbb{C}^* \to \mathbb{C}$	*the $\mathbf{B}$-transform of the function $h : \{\nu \in \mathbb{C} : \lvert\mathrm{Re}(\nu)\rvert \leq \sigma\} \times \mathbb{Z} \to \mathbb{C}$*	(1.9.3)-(1.9.6)
$b(\eta)$, $b(\omega; \ell, q; \eta)$	*(when η is a function in the space $\mathcal{T}^{\ell}_{\sigma}$): a complex constant*	in (6.4.6)
$\mathcal{B}$	*a basis for both $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}$*	below (6.5.26)
$\mathcal{B}_1$	*a basis for $\mathfrak{g}$*	above (6.5.27)
$\mathfrak{C}(\Gamma)$	*a complete set of representatives of the Γ-equivalence classes of cusps*	–
${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$	*a subset of $\mathbb{C}^*$, equal to the domain of the mapping $c \mapsto S_{\mathfrak{a},\mathfrak{b}}(\omega_1, \omega_2; c)$*	in (1.5.9)
$C^{\infty}(G)$	*the space of all smooth functions $f : G \to \mathbb{C}$*	after (1.1.9)
$C^{\infty}(\Gamma\backslash G)$	*the space of all smooth and Γ-automorphic functions on G*	in (1.2.3)
$C^{\infty}(\Gamma\backslash \mathbb{H}_3)$	*the space of all smooth and Γ-automorphic functions on $\mathbb{H}_3$*	in (1.2.15)
$C^{\infty}(N\backslash G, \omega)$	*(when $\omega \in \mathbb{C}$): a certain subspace of $C^{\infty}(G)$*	(1.4.7), (1.4.3)
$C^0(G)$	*the space of all continuous functions $f : G \to \mathbb{C}$*	after (1.1.9)
$C^0(B^+\backslash G)$	*a certain subspace of $C^0(G)$*	start of §6.2
$C^0(N\backslash G, \omega)$	*(when $\omega \in \mathfrak{O}$): a certain subspace of $C^0(B^+\backslash G)$*	start of §6.2
$C^0(\Gamma\backslash G)$	*the space of continuous and Γ-automorphic functions $f : G \to \mathbb{C}$*	–
$c^{\mathfrak{c}}_{V}(\omega)$	*a Fourier coefficient of a cuspidal irreducible subspace V*	in (1.7.13)
$C^{\mathfrak{c}}_{V}(\omega; \nu_V, p_V)$	*a modified Fourier coefficient of a cuspidal subspace*	in (1.7.15)
$\mathrm{d}a$, $\mathrm{d}k$, $\mathrm{d}n$	*left and right Haar measures on the groups A, N and K, respectively*	in (1.1.10)
$\mathrm{d}g$	*a left and right Haar measure on G*	in (1.1.11)
$\mathrm{d}Q$	*a G-invariant measure on $\mathbb{H}_3$*	in (1.1.13)
$D^{\mathfrak{b}}_{\mathfrak{a}}(\omega; \nu, p)$	*a Fourier coefficient of an Eisenstein series*	as in (1.8.4)
$\mathrm{d}_+ z$	*the standard Lebesgue measure on $\mathbb{C}$*	in (1.1.10)
$E^{\mathfrak{a}}_j(q_0, P, K; N, \mathbf{b})$	*a spectral mean, for cusp forms ($j = 0$), or Eisenstein series ($j = 1$)*	(1.9.12), (1.9.13)
$\mathrm{e}(x)$	*equal to* $\exp(2\pi i x)$*, a character for the additive group $\mathbb{R}/\mathbb{Z}$*	–
$\mathcal{E}_{\mathfrak{c}}$	*a 'cusp sector' in $\mathbb{H}_3$*	in (1.1.23)
$(E^{\mathfrak{c}}_{\ell,q}(\nu, p))(g)$	*a Γ-automorphic Eisenstein series associated with the cusp $\mathfrak{c}$*	in §1.8
E	*a subset of $\mathbb{C} \times \mathbb{Z}$ containing all pairs (ν_V, p_V) of spectral parameters*	in (6.7.13)
$\mathcal{F}$, $\mathcal{F}_*$	*fundamental domains for the action of Γ upon $\mathbb{H}_3$*	(1.1.7), (1.1.24)
$(F^{\mathfrak{c}}_m f)(g)$	*the Fourier term of order m for f at $\mathfrak{c}$*	(1.4.1), (1.4.2)
$F_{P,K}(\eta, \xi)$	*a certain polynomial function*	in (4.23)
$g_{\mathfrak{c}}$	*a scaling matrix for the cusp $\mathfrak{c}$*	(1.1.16)-(1.1.21)
G, g	*the special linear group $SL(2, \mathbb{C})$, and one of its elements*	–
$g(a, d; c)$	*a specific element of the group G*	in (6.1.8)
$\mathfrak{g}$	*the complex Lie algebra of G, equal to $\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$*	above (1.2.6)
$h[u]$	$(h[u])_{u \in \mathbb{C}^*}$ is a family of elements of G; $h[u] \in K$ when $\lvert u\rvert = 1$	in (1.1.9)
$\mathbb{H}_3$	*the upper half-space model for three dimensional hyperbolic space*	in §1.1
$\mathbf{H}_2$	*an element of $\mathfrak{k}$ identified with a certain differential operator*	(1.2.9), (1.2.10)
$\mathbf{h}_u$	*(when $u \in \mathbb{C}^*$): a left-translation operator on functions $f : G \to \mathbb{C}$*	(1.5.7), (1.1.9)
$H(\nu, p)$	*an irreducible representation space for $\mathfrak{g}$*	in (1.6.1)
$H^2(\nu, p)$	*a Hilbert space obtained as a certain completion of $H(\nu, p)$*	below (1.6.4)
$\mathcal{H}^{\sigma}_0(\varrho, \vartheta)$	*the space of functions h satisfying conditions (i)-(iii) of Theorem B*	start of §6.7
$\mathcal{H}^{\sigma}_{\star}(\varrho, \vartheta)$	*a certain subspace of $\mathcal{H}^{\sigma}_0(\varrho, \vartheta)$*	above (6.7.1)
$\mathrm{Im}(z)$	*the imaginary part of the complex number z (equal to* $\mathrm{Re}(-iz)$*)*	–

$\mathrm{Int}(U)$	*(if U is a subset of a metric or topological space): the interior of U*	–
$J_\nu(z)$, $J_n(z)$	*a Bessel function*	(1.9.8), (1.9.6)
$J_\nu^*(z)$	*equal to $(z/2)^{-\nu}J_\nu(z)$ when $z>0$*	in (1.9.6)
$\mathcal{J}_{\mu,k}(z)$, $\mathcal{K}_{\nu,p}(z)$	*functions related to Bessel functions of representations of $PSL(2,\mathbb{C})$*	(1.9.5), (1.9.4)
$\mathcal{J}_{\nu,p}^*(z)$	*a function closely related to $\mathcal{J}_{\nu,p}(z)$*	in (6.3.12)
$(\mathbf{J}_\omega\varphi_{\ell,q}(\nu,p))(g)$	*a Jacquet integral*	in §1.5
$\mathbf{J}_\omega$, $\mathbf{J}_\omega^{\nu,p}$	*(when $\omega\in\mathbb{C}$, $\nu\in\mathbb{C}$ and $p\in\mathbb{Z}$): a Jacquet operator*	in §1.5, §1.6
K, $k[\alpha,\beta]$	*the special unitary group, $SU(2)<G$, and one of its elements*	see (1.1.3)
K^+	*a fundamental domain for $\{h[1],h[-1]\}\backslash K$*	–
K-type (ℓ,q)	*classifies elements of $C^\infty(G)$ or $C^\infty(K)$ satisfying certain P.D.E.s*	see (1.3.1)
$\mathfrak{k}$	*the complex Lie algebra of K, equal to $\mathfrak{su}(2)\otimes_{\mathbb{R}}\mathbb{C}$*	above (1.2.9)
$\log(x)$	*equal to $\log_e(x)$, the natural logarithm*	–
$L(s,\chi)$	*Dirichlet's L-series (with Dirichlet character χ)*	–
$L^2(\Gamma\backslash G)$	*the Hilbert space of square-integrable Γ-automorphic functions on G*	(1.2.1), (1.2.2)
$L^2(\Gamma\backslash\mathbb{H}_3)$	*the Hilbert space of square-integrable Γ-automorphic functions on $\mathbb{H}_3$*	(1.2.4), (1.2.5)
${}^0L^2(\Gamma\backslash G)$	*the closure of the subspace of $L^2(\Gamma\backslash G)$ spanned by cusp forms*	below (1.7.3)
${}^{\mathrm{e}}L^2(\Gamma\backslash G)$	*a subspace of $L^2(\Gamma\backslash G)$ spanned by mean values of Eisenstein series*	below (1.7.3)
$L^p(\Gamma\backslash G)$	*a certain space of measurable and Γ-automorphic functions on G*	below (6.5.7)
$L^p(\Gamma\backslash G;\ell,q)$	*a subspace of $L^p(\Gamma\backslash G)$ characterised by the K-type (ℓ,q)*	after (6.5.89)
$L^\infty(\Gamma\backslash G)$	*the space of essentially bounded elements of $L^1(\Gamma\backslash G)$*	above (6.5.6)
$L^\infty(\Gamma\backslash G;\ell,q)$	*a subspace of $L^\infty(\Gamma\backslash G)$ characterised by the K-type (ℓ,q)*	in (6.5.6)
$L^p(N\backslash G)$	*(when $1\le p<\infty$): a certain space of measurable functions on G*	before (6.6.5)
$(\mathbf{L}_{\ell,q}^\omega f)(\nu,p)$	*the Lebedev transform of $f\in P_{\ell,q}(N\backslash G,\omega)$*	in (6.4.2)
$(\widetilde{\mathbf{L}}_{\ell,q}^\omega\eta)(g)$	*the 'inverse Lebedev transform' of $\eta\in\mathcal{T}_\sigma^\ell$*	(6.4.4), (6.4.5)
$(\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta)(g)$	*first modification of the 'inverse Lebedev transform' of $\eta\in\mathcal{T}_\sigma^\ell$*	(6.5.1)-(6.5.3)
$(\widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger}\eta)(g)$	*second modification of the 'inverse Lebedev transform' of $\eta\in\mathcal{T}_\sigma^\ell$*	in (6.5.20)
$m_{\mathfrak{c}}$	*a non-zero Gaussian integer: $\|m_{\mathfrak{c}}\|^2$ is the 'width' of the cusp $\mathfrak{c}$*	below (1.1.22)
$\mathbf{M}_\omega$, $\mathbf{M}_\omega^{\nu,p}$	*the Goodman-Wallach operator on the space $H(\nu,p)$*	(6.3.1), (6.3.2)
N, $n[z]$	*$N=\{n[z]:z\in\mathbb{C}\}<G$*	see (1.1.3)
$\mathfrak{O}$	*equal to $\mathbb{Z}[i]$, the ring of integers of the Gaussian number field $\mathbb{Q}(i)$*	–
P	*the group of those elements of G that are upper triangular matrices*	in (1.1.18)
$\mathbb{P}^1(\mathbb{C})$	*a projective line, identified with the Riemann sphere, $\mathbb{C}\cup\{\infty\}$*	–
$\mathbb{P}^1(\mathbb{Q}(i))$	*a projective line, identified with $\mathbb{Q}(i)\cup\{\infty\}$, the set of all cusps*	above (1.1.16)
$P^{\mathfrak{a}}f$, $P^{\mathfrak{a}}f_\omega$	*a Poincaré series associated with the cusp $\mathfrak{a}$*	in (1.5.4)
$P^{\mathfrak{a},*}\widetilde{\mathbf{L}}_{\ell,q}^\omega\eta$	*a certain pseudo Poincaré series*	in (6.5.5)
$P_{\ell,q}(N\backslash G,\omega)$	*a subspace of $C^\infty(N\backslash G,\omega)$: its elements satisfy growth conditions*	in (6.4.1)
$\underset{\leftrightarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$	*an analytic continuation of $P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$*	(6.5.82)-(6.5.84)
$\underset{\leftrightarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$	*an analytic continuation of $P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$*	see (6.5.89)
q_0	*the 'level' of the Hecke congruence subgroup $\Gamma\le SL(2,\mathfrak{O})$*	(1.1.1), (6.5.96)
$\mathrm{Re}(z)$	*the real part of the complex number z*	–
$\mathcal{R}_{\mathfrak{c}}$	*a compact subset of $\mathbb{C}$ associated with the group $\Gamma_{\mathfrak{c}}$*	in (1.1.22)
$\mathcal{R}(\sigma_1,\sigma_2,t_1)$	*a certain closed rectangular region in the complex plane*	above (6.5.73)
$S(\omega_1,\omega_2;c)$	*a 'simple' (or 'classical') Kloosterman sum*	in (2.16)
$S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c)$	*a generalised Kloosterman sum*	(1.5.8)-(1.5.10)
$T_V\varphi_{\ell,q}(\nu_V,p_V)$	*a Γ-automorphic cusp form on G*	in §1.7
$\mathcal{T}_\sigma^\ell$	*a space of 'test functions' defined on a subset of $\mathbb{C}\times\mathbb{Z}$*	below (6.4.3)
$\mathcal{U}(\mathfrak{g})$	*the universal enveloping algebra of $\mathfrak{g}$*	–
$\mathcal{U}(\mathfrak{k})$	*the universal enveloping algebra of $\mathfrak{k}$*	above (1.2.10)
$U_{\mathfrak{a}}(\psi,c;M;N,b)$	*a sum involving certain sums of Kloosterman sums*	in (1.9.25)
$\mathrm{vol}(\Gamma\backslash G)$	*the covolume of Γ in G*	(1.1.14), (1.1.15)
V	*an irreducible cuspidal subspace of ${}^0L^2(\Gamma\backslash G)$*	below (1.7.4)
$V_{K,\ell,q}$	*a one dimensional subspace of V*	(1.7.5)-(1.7.8)

$W_\omega(\Upsilon;\ell,q)$	*a subspace of* $C^\infty(N\backslash G,\omega)$	in (1.4.8)
$X^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$	*the sum of Kloosterman sums occurring in the spectral sum formula*	in (6.7.2)
$Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$	*equal to the 'spectral side' of the spectral sum formula*	in (6.7.3)
$\mathcal{Z}(\mathfrak{g})$, $\mathcal{Z}(\mathfrak{k})$	*the centres of* $\mathcal{U}(\mathfrak{g})$ *and* $\mathcal{U}(\mathfrak{k})$*, respectively*	–

Other Algebraic Notation. *When U, V and W are groups, the notation $U \leq W$ (resp. $U < W$) is used to indicate that U is a subgroup (resp. proper subgroup) of W. If U and V are subgroups of the group W, then W/V, $U\backslash W$ and $U\backslash W/V$ denote the relevant sets of left cosets, right cosets and double cosets (respectively); and $\left[W:U\right]$ denotes the index of U in W, so that $\left[W:U\right] = |W/U|$. This notation for 'quotients', such as $U\backslash W$ and $U\backslash W/V$, may apply in more general contexts. For example, if U is a subgroup of W, and if S is a subset of the elements of the group W such that $uS \subseteq S$ for all $u \in U$, then S can be expressed as a disjoint union of certain of the right cosets of U in W, and so the notation $U\backslash S$ makes sense (as shorthand for the set of right cosets occurring in that disjoint union). Similar considerations apply in the case of quotients S/V and $U\backslash S/V$, provided that the set S is suitably invariant (either under left-multiplication by elements of $U \leq W$, or under right-multiplication by elements of the group $V \leq W$).*

Other number-theoretic notation. *In relations such as $hm^* \equiv \ell \bmod c\mathfrak{O}$, or in expressions such as the highest common factor (hm^*, c), the rational expression hm^*/c, or (see (2.16)) the 'simple Kloosterman sum' $S(hm^*, \ell; c)$, it is to be understood that m^* denotes an arbitrary element of $\mathfrak{O}$ satisfying $mm^* \equiv 1 \bmod c\mathfrak{O}$. It is therefore implicit in such expressions that one has both $(m,c) \sim 1$ and $(m^*,c) \sim 1$.*

We use the square-brackets notation $[m,n]$ to denote a highest common factor of the Gaussian integers m and n (in the event that m and n are real it should be clear from the context whether or not $[m,n]$ instead denotes a real interval). When ϖ is a Gaussian prime and n is a non-negative integer the relation $\varpi^n \| c$ holds if and only if one has both $\varpi^n \mid c$ and $\varpi^{n+1} \nmid c$.

Summation related conventions. *When a condition of the form '$m \bmod c\mathfrak{O}$' appears below the summation sign, it is to be understood that the variable of summation m ranges (to the extent permitted by any other conditions of summation) over some set of representatives of the cosets of $c\mathfrak{O}$ in $\mathfrak{O}$. Conditions of summation such as $\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma$ (or $\gamma \in \Gamma'_{\mathfrak{a}}\backslash{}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)/\Gamma'_{\mathfrak{b}}$) indicate that the variable of summation γ ranges over some set of representatives of the relevant cosets (or double cosets); this is an abuse of commonly accepted group-theoretic notation, insofar as that in these instances γ does not itself denote a coset, or double coset.*

Where there is no indication to the contrary, variables of summation range over all values in $\mathfrak{O}$ consistent with whatever conditions are attached.

Square roots. *Our use of the square root sign is mildly ambiguous. When the context implies that one has $z > 0$, it is then to be understood that $\sqrt{z}$ denotes the positive square root of z (see (6.6.47) for one such instance); otherwise $\sqrt{z}$ denotes an arbitrary solution $w \in \mathbb{C}$ of the equation $w^2 = z$ (this being the case in (1.9.1), (2.10), (6.1.26), (6.3.11), (6.5.18) and Lemma 6.6.6, for example).*

Notation for bounds and asymptotic estimates. *Where $B \geq 0$, we use the notation $O_{\alpha_1,\ldots,\alpha_n}(B)$ to denote a complex-valued variable β satisfying a condition of the form $|\beta| \leq C(\alpha_1,\ldots,\alpha_n)B$, in which the 'implicit constant' $C(\alpha_1,\ldots,\alpha_n)$ is positive and depends only on previously declared constants and $\alpha_1,\ldots,\alpha_n$. As alternatives to an expression of the form '$\xi = O_{\alpha_1,\ldots,\alpha_n}(B)$', we may prefer to follows Vinogradov in using either '$\xi \ll_{\alpha_1,\ldots,\alpha_n} B$', or '$B \gg_{\alpha_1,\ldots,\alpha_n} \xi$'. Where $A \geq 0$ and $B \geq 0$, the notation $A \asymp_{\alpha_1,\ldots,\alpha_n} B$ may be used to signify that one has both $A \ll_{\alpha_1,\ldots,\alpha_n} B$ and $B \ll_{\alpha_1,\ldots,\alpha_n} A$. There are a few places where, instead of attaching subscripts (to the O, $\ll$, $\gg$ or $\asymp$ sign), we have preferred to explicitly state the parameters upon which the relevant implicit constant may depend.*

By $f(x) = o(\phi(x))$, we mean that the function ϕ is positive valued, and that $f(x)/\phi(x) \to 0$ as x tends to a certain limit: in cases where that limit is not specified it should be understood to be the limit as $x \to \infty$, with $x \in \mathbb{R}$. By $f(x) \sim g(x)$, we mean that $f(x) - g(x) = o(|g(x)|)$.

Measurable sets and functions. *The term 'measurable', when applied to a either a subset of G or a function $f : G \to \mathbb{C}$, is to be understood as meaning that the set, or function, is measurable with respect to the Haar measure $\mathrm{d}g$ on G.*

§2. Lemmas.

In this section are collected the lemmas used in the proof of Proposition 2. Terminology already defined in Section 1 is used freely (without specific references to those definitions).

Lemma 2.1. *Suppose that* $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}' \in \mathbb{Q}(i) \cup \{\infty\}$*, and that* $\tau_1, \tau_2 \in \Gamma = \Gamma_0(q_0) \leq SL(2, \mathfrak{O})$ *satisfy* $\tau_1 \mathfrak{a} = \mathfrak{a}'$*,* $\tau_2 \mathfrak{b} = \mathfrak{b}'$ *(so that* $\mathfrak{a} \stackrel{\Gamma}{\sim} \mathfrak{a}'$ *and* $\mathfrak{b}' \stackrel{\Gamma}{\sim} \mathfrak{b}$*). Let* $g_{\mathfrak{a}}, g_{\mathfrak{b}}, g_{\mathfrak{a}'}, g_{\mathfrak{b}'} \in G = SL(2, \mathbb{C})$ *be chosen so that (1.1.16) and (1.1.20)-(1.1.21) hold for each cusp* $\mathfrak{c} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}'\}$*. Put* $\rho_1 = g_{\mathfrak{a}'}^{-1} \tau_1 g_{\mathfrak{a}}$ *and* $\rho_2 = g_{\mathfrak{b}'}^{-1} \tau_2 g_{\mathfrak{b}}$*. Then, for some* $\beta_1, \beta_2, \eta_1, \eta_2 \in \mathbb{C}$ *with* $\eta_j^2 \in \mathfrak{O}^*$ $(j = 1, 2)$*, one has*

$$\rho_j = \begin{pmatrix} \eta_j & 0 \\ 0 & 1/\eta_j \end{pmatrix} \begin{pmatrix} 1 & \beta_j \\ 0 & 1 \end{pmatrix} \qquad (j = 1, 2), \tag{2.1}$$

$${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}} = \eta_1 \eta_2 {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'} \tag{2.2}$$

and, for $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ *and* $\omega_1, \omega_2 \in \mathfrak{O}$*,*

$$S_{\mathfrak{a},\mathfrak{b}}\left(\omega_1, \omega_2; c\right) = \mathrm{e}\left(\mathrm{Re}\left(\beta_2 \omega_2 - \beta_1 \omega_1\right)\right) S_{\mathfrak{a}',\mathfrak{b}'}\left(\eta_1^{-2} \omega_1 \,,\, \eta_2^{-2} \omega_2 \,;\, \eta_1^{-1} \eta_2^{-1} c\right). \tag{2.3}$$

Proof. The proof is a straightforward adaptation of the proof given on Page 239 of [9] for Equation (1.4) of [9], and is therefore omitted here ■

Lemma 2.2. *If* $\mathfrak{a}$ *is a cusp of* $\Gamma = \Gamma_0(q_0) \leq SL(2, \mathfrak{O})$ *then one has*

$$\mathfrak{a} \stackrel{\Gamma}{\sim} u/w \text{ for some } u, w \in \mathfrak{O} \text{ satisfying } (u, w) \sim 1,\ u \neq 0 \text{ and } w \mid q_0 \,. \tag{2.4}$$

Let $u_1, w_1, u_2, w_2 \in \mathfrak{O}$ *be such that, for* $i = 1, 2$*,* $(u_i, w_i) \sim 1$ *and* $w_i \mid q_0$*. Then* $u_1/w_1 \stackrel{\Gamma}{\sim} u_2/w_2$ *if and only if* $w_2 \sim w_1$ *and* $u_2 w_1/w_2 \equiv \pm u_1 \bmod (w_1, q_0/w_1)\, \mathfrak{O}$*. If* $\mathfrak{C}(\Gamma)$ *is the set of all* Γ*-equivalence classes of cusps then*

$$|\mathfrak{C}(\Gamma)| = \frac{1}{8} \sum_{w \mid q_0} \phi\left((w, q_0/w)\right) + \frac{1}{8} \sum_{\substack{w \mid q_0 \\ (w, q_0/w) \mid 2}} \phi\left((w, q_0/w)\right), \tag{2.5}$$

where ϕ *is Euler's function (i.e.* $\phi(\alpha) = |(\mathfrak{O}/\alpha\mathfrak{O})^*|$ *if* $0 \neq \alpha \in \mathfrak{O}$*).*

Proof. Let $\mathfrak{a}$ be a cusp of $\Gamma = \Gamma_0(q_0)$. Then $\mathfrak{a} \stackrel{\Gamma}{\sim} t/v$ for some $t, v \in \mathfrak{O}$ with $(t, v) \sim 1$. Choose $w \sim (v, q_0)$. Then since $(v/w, q_0/w) \sim 1$ and $(t, w) \sim 1$ it is possible to find a pair $\kappa, \delta \in \mathfrak{O}$ with $(\delta, q_0) \sim 1$ that satisfy the equation $(q_0/w)t\kappa + (v/w)\delta = 1$. One then has $(\delta, q_0\kappa) \sim 1$, so that, for some $\alpha, \beta \in \mathfrak{O}$,

$$\Gamma \ni \begin{pmatrix} \alpha & \beta \\ q_0\kappa & \delta \end{pmatrix} = \gamma \quad \text{(say)}.$$

The result (2.4) follows, for $\gamma(t/v) = (\alpha t + \beta v)/(q_0\kappa t + \delta v) = (\alpha t + \beta v)/w$ where, since $\gamma \in PSL(2, \mathfrak{O})$, one has $(\alpha t + \beta v, w) \sim 1:$ in the event that $\alpha t + \beta v = 0$ one may additionally use $0/1 \stackrel{\Gamma}{\sim} 1/1$ (as $\Gamma_0(q_0) \ni n[1]$).

Consider now the cusps u_i/w_i $(i = 1, 2)$. Supposing they are Γ-equivalent, there are $\alpha, \beta, \kappa, \delta \in \mathfrak{O}$ such that

$$\alpha\delta - q_0\kappa\beta = 1 \qquad \text{and} \qquad \frac{u_2}{w_2} = \frac{\alpha u_1 + \beta w_1}{q_0\kappa u_1 + \delta w_1}\,. \tag{2.6}$$

Since $(u_i, w_i) \sim 1$ $(i = 1, 2)$, the equations in (2.6) imply $w_2 \sim q_0\kappa u_1 + \delta w_1$. Therefore one will have $(w_2, q_0) \sim (\delta w_1, q_0) \sim (w_1, q_0)$, which, as $w_i \,|\, q_0$ for $i = 1, 2$ (by hypothesis), implies the desired conclusion that $w_2 \sim w_1$ when $u_1/w_1 \stackrel{\Gamma}{\sim} u_2/w_2$. From the relations $q_0\kappa u_1 + \delta w_1 \sim w_2 \sim w_1$ (where $w_1 \,|\, q_0$) and (2.6), one may deduce also that $\delta \equiv \varepsilon \bmod (q_0/w_1)\, \mathfrak{O}$ for the same $\epsilon \in \mathfrak{O}^*$ such that $\alpha u_1 + \beta w_1 = \varepsilon u_2 w_1/w_2$. Since the first equation of (2.6) then implies $\alpha \equiv \overline{\varepsilon} \bmod (q_0/w_1)\, \mathfrak{O}$, one obtains:

$$u_2 w_1/w_2 = \overline{\epsilon}\alpha u_1 + \overline{\epsilon}\beta w_1 \equiv (\overline{\epsilon})^2 u_1 \bmod (w_1, q_0/w_1)\, \mathfrak{O}.$$

As $(\overline{\epsilon})^2 = \pm 1$, it has been shown that if u_2/w_2 and u_1/w_1 are Γ-equivalent cusps then $w_2 \sim w_1$ and $u_2 w_1/w_2 \equiv \pm u_1 \bmod (w_1, q_0/w_1)\,\mathfrak{O}$.

In order to establish the converse of what was just found it suffices to consider the case of Gaussian integers u_1, u_2 and w such that

$$(u_1 u_2, w) \sim 1, \quad w\,|\,q_0 \quad \text{and} \quad u_2 \equiv \pm u_1 \bmod (w, q_0/w)\mathfrak{O}\ . \tag{2.7}$$

If in addition $u_2 \equiv u_1 \bmod w\mathfrak{O}$ then it is a straightforward deduction that $u_2/w \overset{\Gamma}{\sim} u_1/w$, for one has $u_2/w = \gamma\,(u_1/w)$ where $\gamma = n\,[(u_2 - u_1)\,/w] \in \Gamma_0(q_0)$.

Now let the assumptions be restricted to (2.7) and the congruence $u_2 \equiv u_1 \bmod (q_0/w)\mathfrak{O}$. For $i = 1, 2$ one can find an element

$$\sigma_i = \begin{pmatrix} u_i & * \\ w & \tilde{u}_i \end{pmatrix} \in SL(2, \mathfrak{O})\ . \tag{2.8}$$

Here $\sigma_i \infty = u_i/w$ $(i = 1, 2)$, so that $u_2/w = \left(\sigma_2 \sigma_1^{-1}\right)(u_1/w)$. Therefore, and since the lower left entry of $\sigma_2 \sigma^{-1}$ is $w\,(\tilde{u}_1 - \tilde{u}_2)$, one sees that if $\tilde{u}_2 \equiv \tilde{u}_1 \bmod (q_0/w)\mathfrak{O}$ then $u_2/w \overset{\Gamma}{\sim} u_1/w$. Given the hypothesis that $u_2 \equiv u_1 \bmod (q_0/w)\mathfrak{O}$ one must have $\tilde{u}_2 \equiv \tilde{u}_1 \bmod (w, q_0/w)\mathfrak{O}$, for (2.8) implies $u_2 \tilde{u}_2 \equiv u_1 \tilde{u}_1 \equiv 1 \bmod w\mathfrak{O}$. Consequently $\tilde{u}_2 = \tilde{u}_1 + \lambda w + \kappa q_0/w$ for some $\lambda, \kappa \in \mathfrak{O}$. On choosing $u_3 = u_1$ and $\tilde{u}_3 = \tilde{u}_1 + \lambda w$, the case $i = 3$ of (2.8) defines a $\sigma_3 \in SL(2, \mathfrak{O})$ such that $u_2/w = \left(\sigma_2 \sigma_3^{-1}\right)(u_1/w)$ and $\sigma_2 \sigma_3^{-1} \in \Gamma_0(q_0)$. This shows that (2.7) and the congruence $u_2 \equiv u_1 \bmod (q_0/w)\mathfrak{O}$ are sufficient to imply $u_2/w \overset{\Gamma}{\sim} u_1/w$. Given the conclusion of the previous paragraph, it is now proven that sufficient conditions for $u_2/w \overset{\Gamma}{\sim} u_1/w$ to hold are that one has (2.7) for the positive choice of sign: alternatively, it would suffice to have (2.7) for the negative choice of sign, since $-u_1/w = h[i]\,(u_1/w)$ and $h[i] \in \Gamma_0(q_0)$.

By the last two paragraphs, sufficient conditions for the Γ-equivalence of u_1/w_1 and u_2/w_2 are that $w_2 \sim w_1$ and $u_2 w_1/w_2 \equiv \pm u_1 \bmod (w_1, q_0/w_1)\,\mathfrak{O}$; the necessity of these conditions had already been established, so (as the lemma claims) one does have $u_1/w_1 \overset{\Gamma}{\sim} u_2/w_2$ if and only if these conditions hold.

The result (2.5) is a direct consequence of the results preceding it in the lemma, with the presence of more than one sum over w being explained by the fact that, when $(u, w) \sim 1$, one will have $-u \equiv u \bmod (w, q_0/w)\mathfrak{O}$ if and only if $(w, q_0/w)\mid 2$ $\blacksquare$

Proposition 2 concerns the special case $\mathfrak{a}' = \mathfrak{a}$ of the generalised Kloosterman sum $S_{\mathfrak{a},\mathfrak{a}'}(\omega, \omega'; c)$ defined in (1.5.8)-(1.5.10). Kloosterman sums of this type are dealt with in the following three lemmas, which are needed for the proof of Proposition 2 (in the next section).

Lemma 2.3. *Suppose that $u, w \in \mathfrak{O}$ and the cusp $\mathfrak{a}'$ of $\Gamma = \Gamma_0(q_0) \leq SL(2, \mathfrak{O})$ are such that one has $(u, w) \sim 1$, $u \neq 0$, $w\,|\,q_0$ and $u/w = \mathfrak{a}'$. Let $v \in \mathfrak{O}$ satisfy*

$$v \sim \frac{q_0}{(q_0, w^2)} \tag{2.9}$$

(so that $|v|^2$ is the 'width' of the cusp $\mathfrak{a}'$, as defined below (1.1.22)) and put

$$g_{\mathfrak{a}'} = \begin{pmatrix} u\sqrt{v} & 0 \\ w\sqrt{v} & (u\sqrt{v})^{-1} \end{pmatrix}. \tag{2.10}$$

Then $g_{\mathfrak{a}'} \in G = SL(2, \mathbb{C})$, and $g_{\mathfrak{a}'}$ is such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} = \mathfrak{a}'$. Suppose moreover that ${}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{a}'}$ and $S_{\mathfrak{a}',\mathfrak{a}'}(\omega, \omega'; c)$ are as given by (1.5.8)-(1.5.10) (i.e. with $\mathfrak{a} = \mathfrak{a}'$ and $g_{\mathfrak{a}} = g_{\mathfrak{a}'}$ there). Then

$${}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{a}'} = \left\{\gamma v w : 0 \neq \gamma \in \mathfrak{O} \text{ and } u\delta^2 + \gamma\delta - u \equiv 0 \bmod (w, q_0/w)\mathfrak{O} \text{ for some } \delta \in \mathfrak{O}\right\} \tag{2.11}$$

and, for $c' = \gamma v w \in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{a}'}$ and $\omega_1, \omega_2 \in \mathfrak{O}$,

$$S_{\mathfrak{a}',\mathfrak{a}'}(\omega_1, \omega_2; c') = \mathrm{e}\!\left(\mathrm{Re}\!\left(\frac{\omega_2 - \omega_1}{uvw}\right)\right) \sideset{}{^*}\sum_{\alpha, \delta \bmod c'\mathfrak{O}} \mathrm{e}\!\left(\mathrm{Re}\!\left(\frac{\omega_1 \alpha + \omega_2 \delta}{c'}\right)\right), \tag{2.12}$$

where the asterisk signifies that the variable of summation, $\delta \bmod c'\mathfrak{O}$, runs over solutions of

$$\delta(u\delta+\gamma) \equiv u \bmod (w, q_0/w)\mathfrak{O} \quad \textit{and} \quad (\delta, \gamma q_0/w) \sim 1 \sim (u\delta+\gamma, w)\ , \tag{2.13}$$

while $\alpha \bmod c'\mathfrak{O}$ is determined by:

$$\alpha\delta \equiv 1 \bmod (\gamma q_0/w)\mathfrak{O} \quad \textit{and} \quad (u_1\alpha-\gamma_1)(u_1\delta+\gamma_1) \equiv u_1^2 \bmod \gamma_1 w\mathfrak{O}\ , \tag{2.14}$$

where $u_1, \gamma_1 \in \mathfrak{O}$ are such that

$$u_1/\gamma_1 = u/\gamma \quad \textit{and} \quad (u_1, \gamma_1) \sim 1\ . \tag{2.15}$$

The congruences (2.14) imply also that $\delta \bmod c'\mathfrak{O}$ is determined by $\alpha \bmod c'\mathfrak{O}$, so that the implied function mapping $\delta \bmod c'\mathfrak{O}$ to $\alpha \bmod c'\mathfrak{O}$ is a bijection from one subset of $\mathfrak{O}/c'\mathfrak{O}$ onto another.

Proof. This lemma is analogous (even in form) to Lemma 2.5 of [9]: the only novelty being that it involves Gaussian integer variables and has $\mathrm{e}(\mathrm{Re}(z))$ in place of $\mathrm{e}(x)$. The proof is such a straightforward adaptation of the proofs of [9], Lemma 2.4 and Lemma 2.5, as to make its inclusion here superfluous ■

The next lemma concerns the more classical type of Kloosterman sum $S(\omega_1, \omega_2; c)$, given by

$$S(\omega_1, \omega_2; c) = \sum_{\substack{\delta \bmod c\mathfrak{O} \\ (\delta, c)\sim 1}} \mathrm{e}\left(\mathrm{Re}\left(\frac{\omega_1\delta^* + \omega_2\delta}{c}\right)\right) \qquad (\omega_1, \omega_2 \in \mathfrak{O} \text{ and } 0 \neq c \in \mathfrak{O}), \tag{2.16}$$

where the dependent variable $\delta^* \bmod c\mathfrak{O}$ is the solution of the congruence $\delta\delta^* \equiv 1 \bmod c\mathfrak{O}$. In it are stated some (almost optimal) upper bounds for $|S(\omega_1, \omega_2; c)|$. These bounds and (2.12) enable one to deduce the bounds on the generalised Kloosterman sum $S_{\mathfrak{a}',\mathfrak{a}'}(\omega_1, \omega_2; c')$ that are contained in Lemma 2.5. Lemma 2.5 is directly analogous to Lemma 2.6 of [9]: another precedent for this type of result may be found in work of Gundlach in Section 4 of [14], which includes what is essentially a 'Weil-Estermann bound' for the analogue of the sum $S_{\mathfrak{a},\mathfrak{b}}(\omega_1, \omega_2; c)$ in the theory of principal congruence subgroups of Hilbert's modular group for any totally real algebraic number field.

Lemma 2.4 (a Weil-Estermann bound over $\mathbb{Q}(i)$). *Let $\omega_1, \omega_2 \in \mathfrak{O}$, $m \in \mathbb{N}$, and suppose that $\varpi \neq 0$ is a prime element of $\mathfrak{O}$. Then*

$$|S(\omega_1, \omega_2; \varpi^m)| \leq \tau_\varpi |\varpi|^{\upsilon_\varpi} |(\omega_1, \omega_2, \varpi^m)\, \varpi^m|\ , \tag{2.17}$$

where $(\tau_\varpi, \upsilon_\varpi) \in \mathbb{R}^2$ is given by

$$(\tau_\varpi, \upsilon_\varpi) = \begin{cases} (8\sqrt{2}, 2) & \textit{if } \varpi\,|\,2\,; \\ (2, 0) & \textit{otherwise.} \end{cases}$$

Moreover, for $0 \neq c \in \mathfrak{O}$, one has

$$|S(\omega_1, \omega_2; c)| \leq 2^{7/2} 2^{\omega(c)} |(\omega_1, \omega_2, c)\, c|\ , \tag{2.18}$$

where $\omega(c)$ is the number of prime ideals of $\mathfrak{O}$ that contain c.

Proof. The bounds (2.17) and (2.18) are special cases of results obtained by Bruggeman and Miatello, in [4], Proposition 9 and Theorem 10 ■

Lemma 2.5. *Let q_0, Γ, $\mathfrak{a}'$, u, v, w, $g_{\mathfrak{a}'}$, ${}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{a}'}$ and the generalised Kloosterman sum $S_{\mathfrak{a}',\mathfrak{a}'}(\omega, \omega'; c')$ be as described in Lemma 2.3. Then $0 \notin {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{a}'} \subset vw\mathfrak{O} \subseteq \mathfrak{O}$ and*

$$|S_{\mathfrak{a}',\mathfrak{a}'}(\omega_1, \omega_2; c')| \leq \sqrt{8}\, |(\omega_1, \omega_2, c')\, c'|\, \tau(c') \qquad \textit{for } \omega_1, \omega_2 \in \mathfrak{O} \textit{ and } c' \in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{a}'}\ , \tag{2.19}$$

where $\tau(c')$ is the number of Gaussian integer divisors of c'.

Proof. Suppose that $c' \in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{a}'}$. Then by the hypotheses, and (2.9) and (2.11) of Lemma 2.3, one has $u, v, w \in \mathfrak{O} - \{0\}$ and $c' = \gamma vw$ for some non-zero $\gamma \in \mathfrak{O}$. It therefore only remains to prove (2.19).

Choose u_1 and γ_1 satisfying the conditions (2.15) of Lemma 2.3, and let $\omega_1, \omega_2 \in \mathfrak{O}$. Then, for $d \in \mathfrak{O}$ with $d \,|\, c'$, let $K(\omega_1, \omega_2; d)$ be given by:

$$K(\omega_1, \omega_2; d) = \sum_{\substack{\alpha,\delta \bmod d\mathfrak{O} \\ \alpha\delta \equiv 1 \bmod (\gamma q_0/w, d)\mathfrak{O} \\ (u_1\alpha - \gamma_1)(u_1\delta + \gamma_1) \equiv u_1^2 \bmod (\gamma_1 w, d)\mathfrak{O}}} \sum \mathrm{e}\left(\mathrm{Re}\left(\frac{\omega_1\alpha + \omega_2\delta}{d}\right)\right). \tag{2.20}$$

It may be deduced from (2.12)-(2.15) of Lemma 2.3 that one has

$$|S_{\mathfrak{a}',\mathfrak{a}'}(\omega_1, \omega_2; c')| = |K(\omega_1, \omega_2; c')| \ . \tag{2.21}$$

A trivial consequence of (2.20) and (2.21) is that $|S_{\mathfrak{a}',\mathfrak{a}'}(\omega_1, \omega_2; c')| \le |c'|^2$; this at least shows that the bound of (2.19) is true for $c' \in \mathfrak{O}^*$; one is therefore to assume henceforth that $c' \in \mathfrak{O}$ is not a unit, or zero. Writing $c' = \varpi_1^{e_1} \cdots \varpi_r^{e_r}$, where $r, e_1, \ldots, e_r \in \mathbb{N}$ and $\varpi_1, \ldots, \varpi_r$ are non-zero prime elements of $\mathfrak{O}$ with $\varpi_i \not\sim \varpi_j$ for $i \ne j$, it follows by the Chinese Remainder Theorem that

$$K(\omega_1, \omega_2; c') = \prod_{j=1}^{r} K\left(\omega_1\lambda_j, \omega_2\lambda_j; \varpi_j^{e_j}\right), \tag{2.22}$$

where, for $j = 1, \ldots, r$, one has $\lambda_j \in \mathfrak{O}$ and $\varpi_j^{-e_j} c' \lambda_j \equiv 1 \bmod \varpi_j^{e_j}\mathfrak{O}$ (so that $(\lambda_j, \varpi_j) \sim 1$).

Let $j \in \{1, \ldots, r\}$ and put $\Omega_k = \omega_k\lambda_j$ $(k = 1, 2)$, $\varpi = \varpi_j$ and $E = e_j$. Then

$$\begin{aligned} K\left(\omega_1\lambda_j, \omega_2\lambda_j; \varpi_j^{e_j}\right) = K\left(\Omega_1, \Omega_2; \varpi^E\right) &= \sum_{\substack{\alpha,\delta \bmod \varpi^E\mathfrak{O} \\ \alpha\delta \equiv 1 \bmod \varpi^F\mathfrak{O} \\ (u_1\alpha - \gamma_1)(u_1\delta + \gamma_1) \equiv u_1^2 \bmod \varpi^G\mathfrak{O}}} \sum \mathrm{e}\left(\mathrm{Re}\left(\frac{\Omega_1\alpha + \Omega_2\delta}{\varpi^E}\right)\right) = \\ &= S_\varpi(E, F, G; \Omega_1, \Omega_2) \qquad \text{(say)}, \end{aligned} \tag{2.23}$$

where F and G are the non-negative integers for which one has

$$\varpi^F \| \gamma q_0/w \qquad \text{and} \qquad \varpi^G \| \gamma_1 w \ . \tag{2.24}$$

By (2.9), (2.15) and the hypothesis that $(u, w) \sim 1$ one has here $[\gamma q_0/w, \gamma_1 w] \sim \gamma q_0/(w, q_0/w) \sim \gamma vw = c'$. Therefore (and since $\varpi^E = \varpi_j^{e_j} \| c'$) it must be the case that

$$\max\{F, G\} = E \in \mathbb{N} \tag{2.25}$$

in (2.23), (2.24). It now suffices to show that

$$|S_\varpi(E, F, G; \Omega_1, \Omega_2)| \le \tau_\varpi |\varpi|^{\upsilon_\varpi} \left|\left(\Omega_1, \Omega_2, \varpi^E\right)\varpi^E\right| \ , \tag{2.26}$$

with $\tau_\varpi \ge 2$ and $\upsilon_\varpi \ge 0$ as defined in Lemma 2.4: for, since one has $\prod_{k=1}^{r} \tau_{\varpi_k} |\varpi_k|^{\upsilon_{\varpi_k}} \le 2^{r+7/2} \le 2^{3/2}\tau(c')$ and $(\lambda_k, \varpi_k) \sim 1$ for $k = 1, \ldots, r$, the equations (2.22), (2.23) and the bound (2.26) directly imply (2.19).

If $G > 0$, then by (2.24) one has $\varpi \nmid u_1$ (since $(u_1, \gamma_1) \sim 1$, $u_1 \,|\, u$ and $(u, w) \sim 1$) and, on choosing $u_1^* \in \mathfrak{O}$ so that $u_1 u_1^* \equiv 1 \bmod \varpi^E\mathfrak{O}$, one may use the linear change of variables of summation given by $\alpha - u_1^*\gamma_1 = \delta_1$, $\delta + u_1^*\gamma_1 = \alpha_1$ to deduce that

$$S_\varpi(E, F, G; \Omega_1, \Omega_2) = \mathrm{e}\left(\mathrm{Re}\left((\Omega_1 - \Omega_2)\gamma_1 u_1^*/\varpi^E\right)\right) S_\varpi(E, G, F; \Omega_2, \Omega_1) \, . \tag{2.27}$$

By (2.25) and the application of (2.27) when $G > F$, one may reduce to considering the cases in which

$$0 \le G \le F = E \in \mathbb{N} \tag{2.28}$$

(it being understood that the values of F and G may have been interchanged here, so that (2.24) might no longer be valid). Proving (2.26) in only these cases will be sufficient, since the condition (2.25) and the upper bound given by (2.26) remain the same if one interchanges F with G, or Ω_1 with Ω_2.

Given (2.28), the congruence conditions on α and δ in the sum in (2.23) simplify to:

$$\alpha\delta \equiv 1 \bmod \varpi^E\mathfrak{O} \quad \text{and} \quad u_1\delta^2 + \gamma_1\delta \equiv u_1 \bmod \varpi^{G_1}\mathfrak{O}\ , \tag{2.29}$$

where $\varpi^{G_1} \sim \varpi^G / \left(\gamma_1, \varpi^G\right)$, so that $0 \leq G_1 \leq G$. When $G_1 = 0$ one therefore has $S_\varpi\left(E,F,G;\Omega_1,\Omega_2\right) = S_\varpi\left(E,E,G;\Omega_1,\Omega_2\right) = S\left(\Omega_1,\Omega_2;\varpi^E\right)$ (the latter term being one of the Kloosterman sums defined by (2.16)), so that by applying the result (2.17) in Lemma 2.4 one obtains exactly the desired bound (2.26).

Suppose now that $G_1 > 0$. Then, since $E > 0$ and $(u_1,\gamma_1) \sim 1$, either it is the case that the congruences in (2.29) have no simultaneous solutions, or else $\varpi \not| u_1$ and the second of those congruences is equivalent to a condition of the form $\delta \equiv \mu \pm \sigma \bmod \varpi^g\mathfrak{O}$, where $\mu,\sigma \in \mathfrak{O}$, $g \in \mathbb{Z}$ and $0 < g \leq G_1$. Therefore either $S_\varpi\left(E,F,G;\Omega_1,\Omega_2\right) = 0$ (in which case (2.26) certainly holds), or for some $\mu,\sigma \in \mathfrak{O}$, and some $g \in \mathbb{Z}$ satisfying $1 \leq g \leq G_1 \leq G \leq F = E$, one has:

$$S_\varpi\left(E,F,G;\Omega_1,\Omega_2\right) = \sum_{\substack{\delta \bmod \varpi^E\mathfrak{O} \\ \delta\equiv\mu\pm\sigma \bmod \varpi^g\mathfrak{O}}} \mathrm{e}\left(\mathrm{Re}\left(\frac{\Omega_1\delta^* + \Omega_2\delta}{\varpi^E}\right)\right) = T_\varpi\left(E,g;\Omega_1,\Omega_2;\mu,\sigma\right) \quad \text{(say)}, \tag{2.30}$$

where δ^*, in the sum over $\delta \bmod \varpi^E\mathfrak{O}$, signifies an element of $\mathfrak{O}$ satisfying $\delta\delta^* \equiv 1 \bmod \varpi^E\mathfrak{O}$ (so that one sums only over δ with $(\delta,\varpi) \sim 1$).

Assuming (2.30) (with $1 \leq g \leq E$), take now H to be the non-negative integer with

$$\varpi^H \sim \left(\Omega_1,\Omega_2,\varpi^E\right) \tag{2.31}$$

and put

$$E_1 = E - H \qquad \text{and} \qquad \Omega_k' = \varpi^{-H}\Omega_k \quad (k = 1,2), \tag{2.32}$$

so that $E_1 \in \mathbb{Z}$, $0 \leq E_1 \leq E$, $\Omega_1',\Omega_2' \in \mathfrak{O}$ and

$$\left(\Omega_1',\Omega_2',\varpi^{E_1}\right) \sim 1\ . \tag{2.33}$$

From (2.30)-(2.32) it follows trivially that

$$\begin{aligned} &\left|S_\varpi\left(E,F,G;\Omega_1,\Omega_2\right)\right| = \left|T_\varpi\left(E,g;\Omega_1,\Omega_2;\mu,\sigma\right)\right| \leq \\ &\qquad \leq 2\left|\varpi^{E-g}\right|^2 = 2|\varpi|^{E+H+E_1-2g} < 2|\varpi|^{E+H} = 2\left|\left(\Omega_1,\Omega_2,\varpi^E\right)\varpi^E\right| \quad \text{if } 2g > E_1\ . \end{aligned} \tag{2.34}$$

By (2.34), one has (2.26) when $2g > E_1$, so (recalling that $g \geq 1$) the only cases requiring further consideration are those in which

$$E_1 \geq 2g \geq 2\ . \tag{2.35}$$

Given (2.31), the summand in the sum appearing in (2.30) is a function of $\delta \bmod \varpi^{E_1}\mathfrak{O}$. Hence and by (2.32) and (2.35) one deduces that

$$T_\varpi\left(E,g;\Omega_1,\Omega_2;\mu,\sigma\right) = \left|\varpi^H\right|^2 T_\varpi\left(E_1,g;\Omega_1',\Omega_2';\mu,\sigma\right). \tag{2.36}$$

Since (2.35) implies $[E_1/2] \geq g \geq 1$, one may adapt the proofs of Salié's formulae for classical Kloosterman sums to prime-power moduli ([21], Lemma 4.1 and Lemma 4.2) so as to obtain here:

$$T_\varpi\left(E_1,g;\Omega_1',\Omega_2';\mu,\sigma\right) = \sum_{\substack{\Delta \bmod \varpi^{E_1}\mathfrak{O} \\ \Delta\equiv\mu\pm\sigma \bmod \varpi^g\mathfrak{O} \\ \Omega_2'\Delta\equiv\Omega_1'\Delta^* \bmod \varpi^{[E_1/2]}\mathfrak{O}}} \mathrm{e}\left(\mathrm{Re}\left(\frac{\Omega_1'\Delta^* + \Omega_2'\Delta}{\varpi^{E_1}}\right)\right), \tag{2.37}$$

where $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\Delta^* \bmod \varpi^{E_1}\mathfrak{O}$ is such that $\Delta\Delta^* \equiv 1 \bmod \varpi^{E_1}\mathfrak{O}$. Since $[E_1/2] \geq 1$, one finds by (2.33) and the last condition of summation in (2.37) that $T_\varpi(E_1, g; \Omega_1', \Omega_2'; \mu, \sigma) \neq 0$ only if $\varpi \not| \Omega_1'\Omega_2'$.

If E_1 is even, or if $\varpi \,|\, 2$, then one requires now no more than a trivial corollary of (2.37):

$$|T_\varpi(E_1, g; \Omega_1', \Omega_2'; \mu, \sigma)| \leq \left|\left\{\Delta \bmod \varpi^{E_1}\mathfrak{O} : \Omega_2'\Delta^2 \equiv \Omega_1' \bmod \varpi^{[E_1/2]}\mathfrak{O}\right\}\right| \leq 2\left|\left(\varpi^2, 2\right)\varpi^{E_1-[E_1/2]}\right|^2 . \tag{2.38}$$

In cases where $(E_1\varpi, 2) \sim 1$ one can follow the method of proof of [21], Lemma 4.2, one step further, to deduce from (2.37) that

$$T_\varpi(E_1, g; \Omega_1', \Omega_2'; \mu, \sigma) = \sum_{\epsilon \in \mathcal{U}} \mathrm{e}\left(2\mathrm{Re}\left(\frac{\Omega_2'\epsilon\Delta_0}{\varpi^{E_1}}\right)\right)\left|\varpi^{[E_1/2]}\right|^2 \sum_{\beta \bmod \varpi\mathfrak{O}} \mathrm{e}\left(\mathrm{Re}\left(\frac{\Omega_2'\epsilon\Delta_0\beta^2}{\varpi}\right)\right), \tag{2.39}$$

where $\mathcal{U}$ is some subset of $\{1, -1\}$ and Δ_0 is (if $\mathcal{U} \neq \emptyset$) a Gaussian integer satisfying $\Omega_2'\Delta_0^2 \equiv \Omega_1' \bmod \varpi^{E_1}$. The innermost sum in (2.39) is an analogue of the classical Gauss sum, and (by a brief elementary manipulation and evaluation of the squared absolute value of the sum) is easily seen to have the same absolute value as ϖ. Therefore, from (2.39) one deduces the upper bound

$$|T_\varpi(E_1, g; \Omega_1', \Omega_2'; \mu, \sigma)| \leq 2|\varpi|^{2[E_1/2]+1} = 2|\varpi|^{E_1} \quad \text{if} \quad (E_1\varpi, 2) \sim 1 . \tag{2.40}$$

On combining (2.36), (2.38) and (2.40) one obtains:

$$|T_\varpi(E, g; \Omega_1, \Omega_2; \mu, \sigma)| \leq |\varpi|^{2H}|T_\varpi(E_1, g; \Omega_1', \Omega_2'; \mu, \sigma)| \leq 2\left|\left(\varpi^2, 2\right)\right|^{5/2}|\varpi|^{H+E} \tag{2.41}$$

in all the cases where (2.35) holds (with H and E_1 given by (2.31), (2.32), so that $H + E_1 = E$); it follows by (2.41), (2.30) and (2.31), that one does have the desired bound (2.26) when (2.35) holds; and this (with the like conclusion having been reached in all other relevant cases) establishes the truth of (2.26) subject to the conditions (2.28); as explained between (2.25) and (2.29), this proof is thereby completed ■

The next result is of a kind long well known in analytic number theory. For examples of closely related results already available in the literature see, for example, the result obtained by Duke in [10], Theorem 1.1, Part (i), and the result obtained by Coleman in [8], Theorem 6.2.

Lemma 2.6. *Let $\alpha, \beta \in \mathbb{R}$, with $\alpha \neq 0$; and let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that*

$$f'(x) = \alpha x^\beta \qquad (x > 0).$$

Suppose also that $0 \neq c \in \mathfrak{O} = \mathbb{Z}[i]$, $a : \mathfrak{O} - \{0\} \to \mathbb{C}$ and $N \geq 1$, and let $s : \mathfrak{O} \times \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ be given by:

$$s(\delta, m, t) = \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} a(\omega)\left(\frac{\omega}{|\omega|}\right)^m \mathrm{e}\left(\mathrm{Re}\left(\frac{\delta\omega}{c}\right) + tf(|\omega|)\right) \qquad (\delta \in \mathfrak{O},\ m \in \mathbb{Z},\ t \in \mathbb{R}).$$

Then, for $M, T > 0$, the sum

$$E_c(N; M, T) = \sum_{\delta \bmod c\mathfrak{O}} \sum_{\substack{m \in \mathbb{Z} \\ |m| \leq M}} \int_{-T}^{T} |s(\delta, m, t)|^2 \mathrm{d}t$$

satisfies

$$E_c(N; M, T) \ll_\beta \left(|c|(M+1) + N^{1/2}\right)\left(|c|T + |\alpha|^{-1}N^{-\beta/2}\right) \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} |a(\omega)|^2.$$

Proof. It is well-known that the real functions

$$\Lambda(x) = \max\{0,\, 1-|x|\} \qquad \text{and} \qquad \operatorname{sinc}^2(x) = \begin{cases} (\pi x)^{-2}\sin^2(\pi x) & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,, \end{cases}$$

are a pair of mutual Fourier transforms, so that

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{t}{T_1}\right) \mathrm{e}(tx)\mathrm{d}t = T_1 \Lambda\left(T_1 x\right) \qquad \text{for } x \in \mathbb{R},\ T_1 > 0\,.$$

They have also the following useful properties:

$$\sum_{m\in\mathbb{Z}} \operatorname{sinc}^2(\Delta m)\mathrm{e}(mx) = \frac{1}{\Delta}\Lambda\left(\frac{\|x\|}{\Delta}\right) \qquad 1 \geq 2\Delta > 0 \text{ and } x \in \mathbb{R}$$

(where $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$);

$$\operatorname{sinc}^2(x) \geq 0 \quad (x \in \mathbb{R}); \quad \text{and} \quad \operatorname{sinc}^2(x) \geq 4\pi^{-2} \quad \text{if } |x| \leq 1/2\,.$$

In combination with the orthogonality of the additive characters $\psi_\delta(\xi) = \mathrm{e}(\mathrm{Re}(\delta\xi/c))$, and the non-negativity of $|s(\delta,m,t)|^2$, the above properties of $\operatorname{sinc}^2(x)$ and $\Lambda(x)$ enable one to see that if $M \geq 1$ and $T > 0$, and if one takes $\Delta = 1/(2M)$ and $T_1 = 2T$, then

$$\begin{aligned} E_c(N;M,T) &\leq \left(\frac{\pi}{2}\right)^4 \sum_{\delta \bmod c\mathfrak{O}} \sum_{m\in\mathbb{Z}} \operatorname{sinc}^2(\Delta m) \int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{t}{T_1}\right) |s(\delta,m,t)|^2 \mathrm{d}t = \\ &= \left(\frac{\pi}{2}\right)^4 \sum_{\substack{\omega_1,\omega_2\in\mathfrak{O} \\ N/2<|\omega_1|^2,|\omega_2|^2\leq N}} a\left(\omega_1\right) \overline{a\left(\omega_2\right)} \prod_{j=1}^{3} F_j\left(\omega_1,\omega_2\right), \end{aligned}$$

where

$$F_1\left(\omega_1,\omega_2\right) = \sum_{\delta \bmod c\mathfrak{O}} \mathrm{e}\left(\mathrm{Re}\left(\frac{\delta\left(\omega_1-\omega_2\right)}{c}\right)\right) = \begin{cases} |c|^2 & \text{if } \omega_1 \equiv \omega_2 \bmod c\mathfrak{O}\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$F_2\left(\omega_1,\omega_2\right) = 2M\Lambda\left(2M\left\|\frac{\mathrm{Arg}\left(\omega_1\right)-\mathrm{Arg}\left(\omega_2\right)}{2\pi}\right\|\right) \quad \text{and} \quad F_3\left(\omega_1,\omega_2\right) = 2T\Lambda\big(2Tf\left(|\omega_1|\right)-2Tf\left(|\omega_2|\right)\big).$$

Therefore, and since $\Lambda(x) = 0$ when $|x| \geq 1$, while $0 \leq \Lambda(x) \leq 1$ for all $x \in \mathbb{R}$, one consequently has:

$$E_c(N;M,T) \leq \frac{\pi^4}{4}|c|^2 MT \sum_{\omega_1,\omega_2} \left|a\left(\omega_1\right)a\left(\omega_2\right)\right|,$$

where the sum is over pairs $(\omega_1,\omega_2) \in \mathfrak{O}\times\mathfrak{O}$ satisfying

$$N/2 < |\omega_1|^2, |\omega_2|^2 \leq N, \qquad \omega_1 \equiv \omega_2 \pmod{c},$$

$$\left\|\frac{\mathrm{Arg}(\omega_1)-\mathrm{Arg}(\omega_2)}{2\pi}\right\| \leq \frac{1}{2M} \quad \text{and} \quad |f(|\omega_1|)-f(|\omega_2|)| \leq \frac{1}{2T}\,.$$

These conditions imply that $\big||\omega_1|-|\omega_2|\big| \leq 2^{|\beta/2|-1}(|\alpha|T)^{-1}N^{-\beta/2}$. Applying the arithmetic-geometric mean inequality to $|a\left(\omega_1\right)|\,|a\left(\omega_2\right)|$, and appealing to symmetry, one finds that

$$E_c(N;M,T) \leq \frac{\pi^4}{4}|c|^2 MT \sum_{N/2<|\omega|^2\leq N} \left|a\left(\omega\right)\right|^2 V_\omega, \tag{2.42}$$

where V_ω is the number of elements $\omega' \in \mathfrak{O}$ satisfying

$$\omega' \equiv \omega \bmod c\mathfrak{O}, \tag{2.43}$$

$$\left\| \frac{\mathrm{Arg}(\omega') - \mathrm{Arg}(\omega)}{2\pi} \right\| \leq \frac{1}{2M} \quad \text{and} \quad \big| |\omega'| - |\omega| \big| \leq \frac{2^{|\beta/2|}}{2|\alpha| T N^{\beta/2}}.$$

The latter two conditions may be simplified to the form

$$r \leq |\omega'| \leq R \quad \text{and} \quad \phi \leq \mathrm{Arg}(\omega') \leq \Phi,$$

where r, R, ϕ, Φ are determined by $\omega, \alpha, \beta, T, M, N$, and satisfy

$$0 \leq r \leq R \leq N^{1/2}, \qquad 0 \leq \phi \leq \Phi \leq 2\pi,$$

$$R - r \ll_\beta \frac{N^{-\beta/2}}{|\alpha| T} \quad \text{and} \quad \Phi - \phi \leq \frac{2\pi}{M}.$$

By this, and by (2.43), the complex numbers $(\omega' - \omega)/c$ are Gaussian integers lying in a simply connected region $\mathcal{R} \subset \mathbb{C}$ with

$$\mathrm{Area}(\mathcal{R}) = \frac{\left(R^2 - r^2\right)(\Phi - \phi)}{2|c|^2} \ll_\beta |\alpha|^{-1} N^{(1-\beta)/2} T^{-1} M^{-1} |c|^{-2}$$

and

$$\mathrm{Perimeter}(\mathcal{R}) = \frac{2(R-r) + (\Phi - \phi)(R + r)}{|c|} \ll_\beta |\alpha|^{-1} N^{-\beta/2} T^{-1} |c|^{-1} + N^{1/2} M^{-1} |c|^{-1} \ .$$

Therefore

$$V_\omega \ll 1 + \mathrm{Perimeter}(\mathcal{R}) + \mathrm{Area}(\mathcal{R}) \ll_\beta \left(1 + N^{1/2} M^{-1} |c|^{-1}\right) \left(1 + |\alpha|^{-1} N^{-\beta/2} T^{-1} |c|^{-1}\right),$$

so that the result of the lemma now follows by (2.42). This completes the proof in cases where $M \geq 1$, and implies the validity, when $M \geq 0$, of a bound for $E_c(N, M+1, T)$ similar to the bound for $E_c(N, M, T)$ appearing in the lemma (and differing only in that the implicit constant may be larger by some factor $b \leq 2$); since one has (trivially) $E_c(N; M+1, T) \geq E_c(N; M, T)$, the remaining cases of the lemma follow ■

Lemma 2.7 (Poisson summation over $\mathbb{Z}^n$ and over $\mathfrak{O} = \mathbb{Z}[i]$). *Let $n \in \mathbb{N}$ and suppose that the function $F : \mathbb{R}^n \to \mathbb{C}$ lies in the Schwartz space: so that, for all $A \geq 0$ and all integers $j, k \geq 0$, the function $\|\mathbf{x}\|^A \frac{\partial^{j_1 + \cdots j_n}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} F(\mathbf{x})$ is continuous and bounded on $\mathbb{R}^n$. Then the Fourier transform*

$$\hat{F}(y_1, \ldots, y_n) = \int_{\mathbb{R}^n} F(\mathbf{x}) \mathrm{e}(-\mathbf{x} \cdot \mathbf{y}) \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{2.44}$$

is a complex-valued functions defined on $\mathbb{R}^n$, and lies in the Schwartz space. One has, moreover:

$$\sum_{\mathbf{m} \in \mathbb{Z}^n} F(\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \hat{F}(\mathbf{m}) \ . \tag{2.45}$$

When $n = 2$ and $f : \mathbb{C} \to \mathbb{C}$ is given by $f(x + iy) = F(x, y)$ for $(x, y) \in \mathbb{R}^2$ (with $F(\mathbf{x})$ as above), the complex Fourier transform

$$\hat{f}(w) = \int_{\mathbb{C}} f(z) \mathrm{e}\left(-\mathrm{Re}(wz)\right) \mathrm{d}_+ z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x + iy) \mathrm{e}\left(-\mathrm{Re}((x + iy)w)\right) \mathrm{d}x \mathrm{d}y \tag{2.46}$$

is a complex-valued function defined on $\mathbb{C}$, and one has

$$\sum_{\alpha\in\mathfrak{O}} f(\alpha) = \sum_{\alpha\in\mathfrak{O}} \hat{f}(\alpha) \ . \tag{2.47}$$

Proof. For the results up to and including (2.45) see, for example [30], Chapter 13, Section 4 and Section 6. The results (2.46) and (2.47) amount to a special case of (2.44)-(2.45), for the right-hand side of (2.46) is (by definition) equal to $\hat{F}\left(\mathrm{Re}(w), -\mathrm{Im}(w)\right)$ and, since complex conjugation is a permutation of $\mathfrak{O}$, it therefore follows from (2.45) that $\sum_{\alpha\in\mathfrak{O}} f(\alpha) = \sum_{\alpha\in\mathfrak{O}} \hat{f}(\overline{\alpha}) = \sum_{\alpha\in\mathfrak{O}} \hat{f}(\alpha)$ ∎

Lemma 2.8. *Let f and F be as in the case $n = 2$ of Lemma 2.7. For $z = x + iy$ with $x, y \in \mathbb{R}$ put*

$$(\boldsymbol{\Delta}_{\mathbb{C}} f)(z) = (\boldsymbol{\Delta}_{\mathbb{R}\times\mathbb{R}} F)(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x+iy) \ .$$

Then $\boldsymbol{\Delta}_{\mathbb{R}\times\mathbb{R}} F$ (the Laplacian of F) is a member of the Schwartz space. The functions f and $\boldsymbol{\Delta}_{\mathbb{C}} f$ have Fourier transforms $\hat{f}, \widehat{\boldsymbol{\Delta}_{\mathbb{C}} f} : \mathbb{C} \to \mathbb{C}$ (defined as in Lemma 2.7), which are related to one another by:

$$|2\pi w|^2 \hat{f}(w) = -\widehat{\boldsymbol{\Delta}_{\mathbb{C}} f}(w) \quad \text{for } w \in \mathbb{C}. \tag{2.48}$$

For all $w \in \mathbb{C} - \{0\}$ and $j = 0, 1, 2, \ldots$, one has

$$\left|\hat{f}(w)\right| = (2\pi|w|)^{-2j} \left|\widehat{\boldsymbol{\Delta}^j_{\mathbb{C}} f}(w)\right| \leq (2\pi|w|)^{-2j} \left|\widehat{\boldsymbol{\Delta}^j_{\mathbb{C}} f}\right|(0) = (2\pi|w|)^{-2j} \int_{\mathbb{C}} \left|\left(\boldsymbol{\Delta}^j_{\mathbb{C}} f\right)(z)\right| \mathrm{d}_+ z \ . \tag{2.49}$$

Proof. Since F is a member of the Schwartz space, so is $\left(\partial^2/\partial x^2\right)F(x,y)$. Therefore two applications of integration by parts suffice to show that, when $u, y \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \mathrm{e}(-ux) F_{11}(x,y) \mathrm{d}x = (2\pi i u)^2 \int_{-\infty}^{\infty} F(x,y) \mathrm{e}(-ux) \mathrm{d}x \ ,$$

where $F_{11}(x,y) = (\partial/\partial x)^2 F(x,y)$. It follows that $\hat{F}_{11}(u,v) = -(2\pi u)^2 \hat{F}(u,v)$, for $u, v \in \mathbb{R}$. Similarly, integrations by parts with respect to y yield $\hat{F}_{22}(u,v) = -(2\pi v)^2 \hat{F}(u,v)$, where $F_{22}(x,y) = (\partial/\partial y)^2 F(x,y)$. Therefore, for $u, v \in \mathbb{R}$ and $w = u + iv$, one has:

$$|2\pi w|^2 \hat{f}(w) = \left((2\pi u)^2 + (2\pi v)^2\right) \hat{F}(u,-v) = -\left(\hat{F}_{11}(u,-v) + \hat{F}_{22}(u,-v)\right) = -\widehat{\boldsymbol{\Delta}_{\mathbb{R}\times\mathbb{R}} F}(u,-v) = -\widehat{\boldsymbol{\Delta}_{\mathbb{C}} f}(w) \ ,$$

which is (2.48). Since $\boldsymbol{\Delta}_{\mathbb{C}}\, \mathrm{e}\big(\mathrm{Re}(wz)\big) = -|2\pi w|^2 \mathrm{e}\big(\mathrm{Re}(wz)\big)$, the result (2.48) is (in essence) illustrative of the fact that the Laplacian operator for functions on $\mathbb{R}^n$ is symmetric on a dense subspace of the Lebsgue space $L^2\left(\mathbb{R}^n\right)$.

From (2.48) it follows by induction that

$$\hat{f}(w) = (-1)^j |2\pi w|^{-2j} \, \widehat{\boldsymbol{\Delta}^j_{\mathbb{C}} f}(w) \quad \text{for } w \in \mathbb{C} - \{0\} \text{ and } j = 0, 1, 2, \ldots \ .$$

This implies the first equality of (2.49): the rest of (2.49) follows since (by virtue of $\left(\boldsymbol{\Delta}^j_{\mathbb{R}\times\mathbb{R}} F\right)(x,y)$ being a member of the Schwartz space, and as $|\mathrm{e}(x)| = 1$ for $x \in \mathbb{R}$) it is the case that the function $g(z) = \left(\boldsymbol{\Delta}^j_{\mathbb{C}} f\right)(z) \mathrm{e}(-\mathrm{Re}(wz))$ satisfies

$$\left|\int_{\mathbb{C}} g(z)\, \mathrm{d}_+ z\right| \leq \int_{\mathbb{C}} |g(z)|\, \mathrm{d}_+ z = \int_{\mathbb{C}} \left|\left(\boldsymbol{\Delta}^j_{\mathbb{C}} f\right)(z)\right| \mathrm{d}_+ z < +\infty \quad ∎$$

§3. The proof of Proposition 2.

Part I: a proof of (1.9.26) using the Weil-Estermann bound. By Lemma 2.2 one may choose coprime non-zero Gaussian integers u, w such that $w \mid q_0$ and $u/w \stackrel{\Gamma}{\sim} \mathfrak{a}$. One then has $\tau\mathfrak{a} = u/w$ for some $\tau \in \Gamma$. Choose next a Gaussian integer v satisfying hypothesis (2.9) of Lemma 2.3. Then, on putting $\rho = g_{u/w}^{-1}\tau g_{\mathfrak{a}}$, where $g_{u/w}$ is the scaling matrix $g_{\mathfrak{a}'}$ given by (2.10) of Lemma 2.3, it follows by Lemma 2.1 that $\rho = h[\eta]n[\beta]$ for some $\beta \in \mathbb{C}$ and some $\eta \in \mathbb{C}$ with $\eta^2 \in \mathfrak{O}^*$, and that this β and the unit $\eta^2 = \epsilon$ (say) are such that one has both

$$ {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}} = \epsilon\,\mathcal{C}' , \quad \text{where} \quad \mathcal{C}' = {}^{u/w}\mathcal{C}^{u/w} , \tag{3.1} $$

and, when $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}$ and $\omega_1, \omega_2 \in \mathfrak{O}$,

$$ S_{\mathfrak{a},\mathfrak{a}}\left(\omega_1, \omega_2; c\right) = \mathrm{e}\left(\mathrm{Re}\left(\beta\left(\omega_2 - \omega_1\right)\right)\right) S_{u/w,u/w}\left(\frac{\omega_1}{\epsilon}\,, \frac{\omega_2}{\epsilon}\,; \frac{c}{\epsilon}\right) \tag{3.2} $$

(note that the case of Lemma 2.1 used here is that in which $\mathfrak{b} = \mathfrak{a}$, $\mathfrak{b}' = \mathfrak{a}' = u/w$, $g_{\mathfrak{b}} = g_{\mathfrak{a}}$, $g_{\mathfrak{b}'} = g_{\mathfrak{a}'} = g_{u/w}$ and $\tau_1 = \tau_2 = \tau$: one therefore has $\rho_1 = \rho_2 = \rho$, which, by (2.1), implies both $\eta_1 = \eta_2 = \eta$, say, and $\beta_1 = \beta_2 = \beta$, say). In view of Lemma 2.5 (where $\mathfrak{a}' = u/w$), it follows from (3.1), (2.9) and (3.2) that

$$ 0 \notin {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}} \subset \epsilon v w \mathfrak{O} = v w \mathfrak{O} = \frac{1}{\mu(\mathfrak{a})}\,\mathfrak{O} \tag{3.3} $$

(where $\mu(\mathfrak{a})$ is as discussed in Remark 1.9.7, below Theorem 1), and that

$$ \left|S_{\mathfrak{a},\mathfrak{a}}\left(\omega_1, \omega_2; c\right)\right| \leq \sqrt{8}\left|\left(\frac{\omega_1}{\epsilon}\,, \frac{\omega_2}{\epsilon}\,, \frac{c}{\epsilon}\right)\frac{c}{\epsilon}\right| \tau\left(\frac{c}{\epsilon}\right) = \sqrt{8}\left|\left(\omega_1\omega_2, c\right) c\right| \tau(c) , \tag{3.4} $$

for $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}$ and $\omega_1, \omega_2 \in \mathfrak{O}$. Note that (3.3) is the result (1.9.24) of the proposition. Applying (3.4) directly to (1.9.25), one obtains (subject to the hypotheses of the proposition) the upper bound:

$$ \left|U_{\mathfrak{a}}(\psi, c; M; N, b)\right| \leq \sqrt{8}\,(2M+1) \sum_{\substack{\omega_1,\omega_2 \in \mathfrak{O} \\ N/2 < |\omega_1|^2, |\omega_2|^2 \leq N}} \left|b\left(\omega_1\right) b\left(\omega_2\right)\left(\omega_1\omega_2, c\right) c\right| \tau(c) . $$

This bound for $|U_{\mathfrak{a}}(\psi, c; M; N, b)|$ implies the result in (1.9.26): for one has, by the Cauchy-Schwarz inequality,

$$ \sum_{\substack{\omega_1,\omega_2 \in \mathfrak{O} \\ N/2 < |\omega_1|^2, |\omega_2|^2 \leq N}} \left|\left(\omega_1, \omega_2, c\right) b\left(\omega_1\right) b\left(\omega_2\right)\right| \leq \left(E(c,N)\right)^{1/2} \left\|b_N\right\|_2^2 , $$

where $\|b_N\|_2^2$ is as defined in (1.9.16), while

$$ \begin{aligned} E(c,N) = & \sum_{\substack{\omega_1,\omega_2 \in \mathfrak{O} \\ N/2 < |\omega_1|^2, |\omega_2|^2 \leq N}} \left|\left(\omega_1, \omega_2, c\right)\right|^2 \leq \\ & \leq \frac{1}{4}\sum_{\delta \mid c} |\delta|^2 \Bigg(\sum_{\substack{\omega \in \delta\mathfrak{O} \\ 0 < |\omega|^2 \leq N}} 1\Bigg)^2 = \sum_{\delta \mid c} |\delta|^2 \left(O\left(\frac{N}{|\delta|^2}\right)\right)^2 \ll N^2 \sum_{\delta \mid c} \frac{1}{|\delta|^2} \leq N^2 \tau(c) \quad \square \end{aligned} $$

Part II: a proof of (1.9.27) by direct use of Lemma 2.6. The bounds (1.9.27) and (1.9.28) remain to be proved. Bounds on the absolute values of the relevant Kloosterman sums are not sufficient to achieve this: one may use instead the explicit representation given by (2.12) of Lemma 2.3. Bearing in mind (3.1)-(3.3), the result (2.12) shows that

$$ S_{\mathfrak{a},\mathfrak{a}}\left(\omega_1, \omega_2; c\right) = \mathrm{e}\left(\mathrm{Re}\left(\epsilon^{-1}\beta'\left(\omega_2 - \omega_1\right)\right)\right) \sideset{}{^*}\sum_{\alpha,\delta \bmod c'\mathfrak{O}} \mathrm{e}\left(\mathrm{Re}\left(\frac{\epsilon^{-1}\omega_1\alpha + \epsilon^{-1}\omega_2\delta}{c'}\right)\right) , $$

where

$$\begin{aligned} \beta' &= \epsilon\beta + \frac{1}{uvw}\,, \\ c' &= c/\epsilon \in \mathcal{C}' \subseteq \frac{1}{\mu(\mathfrak{a})}\,\mathfrak{O} - \{0\}\,, \end{aligned} \tag{3.5}$$

and where the asterisk indicates the same conditions of summation as in (2.13)-(2.15) of Lemma 2.3. By using this to rewrite the right-hand side of (1.9.25), then substituting $\epsilon\omega_1$ and $\epsilon\omega_2$ for ω_1 and ω_2 (respectively) and noting that $|\epsilon|^{2m} = 1^{2m} = 1$, one obtains:

$$U_{\mathfrak{a}}(\psi, c; M; N, b) = \sum_{m=-M}^{M} \Bigg| \sum_{\substack{\omega_1,\omega_2 \in \mathfrak{O} \\ N/2<|\omega_1|^2,|\omega_2|^2\le N}} \overline{b\left(\epsilon\omega_1\right)} b\left(\epsilon\omega_2\right) \left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^m \mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c'|} + \mathrm{Re}\left(\beta'\left(\omega_2-\omega_1\right)\right)\right) \times$$

$$\times \sum_{\alpha,\delta \bmod c'\mathfrak{O}}^{*} \mathrm{e}\left(\mathrm{Re}\left(\frac{\omega_1\alpha+\omega_2\delta}{c'}\right)\right) \Bigg| =$$

$$= \sum_{m=-M}^{M} \Bigg| \sum_{\substack{\omega_1,\omega_2 \in \mathfrak{O} \\ N/2<|\omega_1|^2,|\omega_2|^2\le N}} \overline{a(\omega_1)} a(\omega_2) \left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^m \mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c'|}\right) S^*(\omega_1,\omega_2;c') \Bigg| \,, \tag{3.6}$$

where

$$a(\omega) = b(\epsilon\omega)\,\mathrm{e}(\mathrm{Re}\left(\beta'\omega\right)) \qquad (0 \neq \omega \in \mathfrak{O}) \tag{3.7}$$

and

$$S^*(\omega_1,\omega_2;c') = \sum_{\alpha,\delta \bmod c'\mathfrak{O}}^{*} \mathrm{e}\left(\mathrm{Re}\left(\frac{\omega_1\alpha+\omega_2\delta}{c'}\right)\right) \qquad (\omega_1,\omega_2 \in \mathfrak{O}) \tag{3.8}$$

with the asterisk superfixed to the summation sign having the same meaning as in Lemma 2.3. It will therefore suffice to bound the slightly more general sum:

$$U^{\circ}_{q_0,w,u}\left(\psi, c'; M, N\right) = \sum_{m=-M}^{M} \eta_m \sum_{\substack{\omega_1,\omega_2 \in \mathfrak{O} \\ N/2<|\omega_1|^2,|\omega_2|^2\le N}} \overline{a(\omega_1)} a(\omega_2) \left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^m \mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c'|}\right) S^*(\omega_1,\omega_2;c')\,, \tag{3.9}$$

where the coefficients η_m are arbitrary complex numbers satisfying

$$|\eta_m| = 1 \quad \text{for} \quad m = -M, -M+1, \ \ldots\ , M\,. \tag{3.10}$$

The next step requires the introduction of a ‘redundant weighting function’ $f : (0,\infty) \to [0,1]$ possessing a continuous second derivative, and such that

$$f(x) = \begin{cases} 0 & \text{if } x \le 1/2 \text{ or } x \ge 2\,; \\ 1 & \text{if } 1/\sqrt{2} \le x \le 1\,. \end{cases}$$

Such a function can be explicitly defined (if need be). Here it will suffice to suppose that f is chosen absolutely independently of all factors other than the above explicit requirements: by this supposition there exists an absolute constant $C_2 \in [1,\infty)$ such that $\left|f^{(j)}(x)\right| \le C_2$ for $j = 0,1,2$ and $x \in \mathbb{R}$. For this f one has (see Appendix A.2 of [19]):

$$f(x)\mathrm{e}\left(\frac{\psi\sqrt{N}\,x}{|c'|}\right) = \int_{-\infty}^{\infty} g_Y(t) x^{it}\,\mathrm{d}t \qquad (x > 0), \tag{3.11}$$

where

$$Y = \frac{\psi\sqrt{N}}{|c'|} \in \mathbb{R} \tag{3.12}$$

and $g_Y : \mathbb{R} \to \mathbb{C}$ is the Mellin-transform given by

$$g_Y(t) = \frac{1}{2\pi}\int_0^\infty f(x)\mathrm{e}\,(Yx)\,x^{-1-it}\mathrm{d}x\ .$$

The point of these observations is that, since $f\left(\sqrt{|\omega_1\omega_2|\,/N}\,\right) = 1$ when $N/2 < |\omega_1|^2, |\omega_2|^2 \leq N$, the case $x = \sqrt{|\omega_1\omega_2|\,/N}$ of (3.11) therefore provides a means to effect a 'separation of variables' in the summand on the right-hand side of equation (3.9) (the relevant variables being ω_1 and ω_2).

Practical application of (3.11) requires suitable estimates for $|g_Y(t)|$. One has, trivially,

$$g_Y(t) \ll \int_0^\infty |f(x)|x^{-1}\mathrm{d}x \leq \int_{1/2}^2 x^{-1}\mathrm{d}x \ll 1\ .$$

For less trivial estimates, in cases where $t \neq 0$, it is helpful to note that

$$g_Y(t) = \int_{1/2}^2 F(x)e^{iu(x)}\mathrm{d}x \tag{3.13}$$

where

$$F(x) = (2\pi x)^{-1}f(x) \quad \text{and} \quad u(x) = u_{Y,t}(x) = 2\pi Yx - t\log x.$$

Here $\inf_{1/2\leq x\leq 2}|u''(x)| = \inf_{1/2\leq x\leq 2}|t|/x^2 \geq |t|/4$, while the choice of f ensures that $\sup_{1/2\leq x\leq 2}|F(x)| \ll 1$ and $\int_{1/2}^2 |F'(x)|\,\mathrm{d}x \ll 1$; it therefore follows by the 'second derivative test' (as formulated in Lemma 5.1.3 of [18]) that one has, uniformly with respect to Y,

$$g_Y(t) \ll |t|^{-1/2} \qquad \text{for } 0 \neq t \in \mathbb{R}.$$

A third bound on $g_Y(t)$ is useful in cases where

$$|t| \geq T_0 = 8\pi|Y|. \tag{3.14}$$

In such cases one finds that

$$\frac{|t|}{4} \leq \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| = \left|2\pi Y - \frac{t}{x}\right| \leq 3|t| \qquad (1/2 < x < 2),$$

so that one may rewrite (3.13) as

$$g_Y(t) = \int_a^b F(x)\frac{\mathrm{d}x}{\mathrm{d}u}\,e^{iu}\mathrm{d}u,$$

with $|b-a| < 6|t|$ ($a = u(1/2)$, $b = u(2)$). Then two integrations-by-parts show that

$$g_Y(t) = -\int_a^b \left(F''(x)\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^3 + 3F'(x)\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}^2x}{\mathrm{d}u^2} + F(x)\frac{\mathrm{d}^3x}{\mathrm{d}u^3}\right)e^{iu}\mathrm{d}u$$

(boundary terms are absent, since f is supported in the interval $[1/2, 2]$, and has a continuous second derivative on $(-\infty, \infty)$). Here it follows by elementary real-variable calculus that when $a < u < b$ (so that $1/2 < x < 2$) one has

$$F^{(j-1)}(x) \ll 1 \qquad \text{and} \qquad \frac{\mathrm{d}^jx}{\mathrm{d}u^j} \ll |t|^{-j} \qquad (j = 1, 2, 3).$$

Consequently one deduces from the last equation involving $g_Y(t)$ that if (3.14) holds then

$$g_Y(t) \ll |b-a||t|^{-3} \ll |t|^{-2} \ .$$

On combining this last bound for $g_Y(t)$ with the others found before, one has:

$$g_Y(t) \ll \begin{cases} 1 & \text{if } |t| \le 1; \\ |t|^{-1/2} & \text{if } 1 < |t| \le 1 + T_0; \\ |t|^{-2} & \text{if } |t| > 1 + T_0. \end{cases} \tag{3.15}$$

In light of the final remark in the paragraph containing (3.11), one may attach to each summand on the right-hand side of Equation (3.9) an extra factor $f(\sqrt{|\omega_1\omega_2|/N})$ (this action does not change the value of any summand in (3.9), for it is equivalent in effect to multiplying each summand by 1). Hence, by applying the case $x = \sqrt{|\omega_1\omega_2|\,/N}$ of (3.11) and recalling the definition (3.8) of $S^*(\omega_1,\omega_2;c')$, one finds that

$$U^{\circ}_{q_0,w,u}\left(\psi, c'; M, N\right) = \int_{-\infty}^{\infty} g_Y(t)\sigma_{q_0,w,u}\left(c'; M, N; t\right) N^{-it/2}\mathrm{d}t,$$

where

$$\sigma_{q_0,w,u}\left(c'; M, N; t\right) = \sum_{\alpha,\delta \bmod c'\mathfrak{O}}{}^{*} \sum_{m=-M}^{M} \eta_m A\left(\frac{\alpha}{c'}, m, t\right) \overline{A\left(-\frac{\delta}{c'}, -m, -t\right)} \ ,$$

with

$$A(\theta, n, v) = \sum_{\substack{\omega\in\mathfrak{O} \\ N/2<|\omega|^2\le N}} \overline{a\left(\omega\right)}\, \mathrm{e}\left(\mathrm{Re}\left(\omega\theta\right)\right) \left(\frac{\omega}{|\omega|}\right)^{n} |\omega|^{iv/2} \ . \tag{3.16}$$

From this and the estimates in (3.15) one can deduce that

$$U^{\circ}_{q_0,w,u}\left(\psi, c'; M, N\right) \ll \int_1^{2(1+T_0)} T^{-3/2}\mathcal{E}(T)\mathrm{d}T + \int_{2(1+T_0)}^{\infty} T^{-3}\mathcal{E}(T)\mathrm{d}T \ , \tag{3.17}$$

where, by (3.10) and the arithmetic-geometric mean inequality,

$$\mathcal{E}(T) = \int_{-T}^{T} \left|\sigma_{q_0,w,u}\left(c'; M, N; t\right)\right|\mathrm{d}t \le \frac{1}{2}\int_{-T}^{T} \sum_{\alpha,\delta \bmod c'\mathfrak{O}}{}^{*} \sum_{m=-M}^{M} \left(\left|A\left(\frac{\alpha}{c'}, m, t\right)\right|^2 + \left|A\left(-\frac{\delta}{c'}, m, t\right)\right|^2\right)\mathrm{d}t \ .$$

In this last upper bound the variable α is dependent upon the variable of summation δ, and the conditions of summation (on both δ and α) are those described in (2.13)-(2.15) of Lemma 2.3 (note already having been made of this below (3.8)). Since the very last part of Lemma 2.3 implies that the the function mapping $\delta \bmod c'\mathfrak{O}$ to $\alpha \bmod c'\mathfrak{O}$ is injective (as is the function mapping $\delta \bmod c'\mathfrak{O}$ to $-\delta \bmod c'\mathfrak{O}$), one is therefore able to deduce from the last bound for $\mathcal{E}(T)$ that

$$\mathcal{E}(T) \le \int_{-T}^{T} \sum_{\lambda \bmod c'\mathfrak{O}} \sum_{m=-M}^{M} \left|A\left(\frac{\lambda}{c'}, m, t\right)\right|^2 \mathrm{d}t \ .$$

Hence (and by (3.16)) an appeal to the case $\alpha = (4\pi)^{-1}$, $\beta = -1$ of Lemma 2.6 yields the bound:

$$\mathcal{E}(T) \ll \left(|c'|\,(M+1) + N^{1/2}\right)\left(|c'|\,T + N^{1/2}\right) \sum_{\substack{\omega\in\mathfrak{O} \\ N/2<|\omega|^2\le N}} |a(\omega)|^2 \qquad (T \ge 1).$$

It follows by (3.17), the equality in (3.14), (3.12), (3.7) and (3.5) (in which $\epsilon \in \mathfrak{O}^*$) that one has:

$$\begin{aligned} U^{\circ}_{q_0,w,u}(\psi, c'; M, N) &\ll \left(|c|\,(M+1) + N^{1/2}\right)\left(|c|\,(1+T_0)^{1/2} + N^{1/2}\right)\|\mathbf{b}_N\|_2^2 \asymp \\ &\asymp \left(|c|\,(M+1) + N^{1/2}\right)\left(|c| + |c|^{1/2} N^{1/4}|\psi|^{1/2} + N^{1/2}\right)\|\mathbf{b}_N\|_2^2 \leq \\ &\leq \frac{\sqrt{5}}{2}\,(1+|\psi|)^{1/2}\left(|c|\,(M+1) + N^{1/2}\right)\left(|c| + N^{1/2}\right)\|\mathbf{b}_N\|_2^2\ , \end{aligned}$$

where the $\|\mathbf{b}_N\|_2$ notation is as defined in (1.9.16). This (given (3.6), (3.9) and (3.10)) completes the proof of the result (1.9.27) of the proposition $\square$

Part III: a proof of (1.9.30) by use of the Cauchy-Schwarz inequality and Lemma 2.6. It now only remains to prove the conditional bound (1.9.30): so suppose now that c and ψ satisfy the additional constraints in (1.9.28) and (1.9.29). Then, on setting

$$L = \min\left\{M\ ,\ \left[\frac{N^{(1-\varepsilon)/2}}{|c'|}\right]\right\} \in \mathbb{N}\cup\{0\} \tag{3.18}$$

(with c' as indicated by (3.5) and (3.1)-(3.3)), it follows by subdivision of the outer sum in (3.6) (and positivity of the terms in this sum) that, for some pair of integers M_1, M_2 satisfying both $-M \leq M_1 \leq M_2 \leq M$ and $M_2 - M_1 = 2L$, one has:

$$\begin{aligned} U_{\mathfrak{a}}(\psi, c; M; N, b) &\ll \left(\frac{M+1}{L+1}\right)\sum_{m=M_1}^{M_2}\left|\ \sum\sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\leq N}} \overline{a(\omega_1)}\,a(\omega_2)\left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^m \mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c'|}\right)S^*(\omega_1,\omega_2;c')\right| = \\ &= \left(\frac{M+1}{L+1}\right)U^{\star}_{q_0,w,u}(\psi, c'; M_1, M_2; N) \qquad\qquad \text{(say).} \end{aligned} \tag{3.19}$$

After substituting $m + L + M_1$ for m, one finds that one has in (3.19):

$$U^{\star}_{q_0,w,u}(\psi, c'; M_1, M_2; N) = \sum_{m=-L}^{L}\eta'_m \sum\sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\leq N}} \overline{a^-(\omega_1)}\,a^+(\omega_2)\left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^m \mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c'|}\right)S^*(\omega_1,\omega_2;c')\,,$$

where

$$a^{\pm}(\omega) = a(\omega)\left(\frac{\omega}{|\omega|}\right)^{\pm(L+M_1)} \qquad\qquad (0\neq\omega\in\mathfrak{O}) \tag{3.20}$$

and the coefficients η'_m are certain complex numbers satisfying

$$|\eta'_m| = 1 \qquad \text{for } m = -L, -L+1,\ \ldots\ , L\,. \tag{3.21}$$

Using a change in the order of summation, and a subdivision of the sums over both m and ω_2, it may now be deduced that, for each $H\in\mathbb{N}$, there exists some positive integer $h = h(H) \leq H$ such that

$$\begin{aligned} \left|U^{\star}_{q_0,w,u}(\psi, c'; M_1, M_2; N)\right| \leq H &\left|\sum_{r=0}^{1}\ \sum_{\substack{\omega_1\in\mathfrak{O}\\ N/2<|\omega_1|^2\leq N}} \overline{a^-(\omega_1)}\ \times\right. \\ &\left.\times \sum_{\substack{\omega_2\in\mathfrak{O}\\ \mathcal{N}(h)<|\omega_2|^2\leq\mathcal{N}(h-1)}} a^+(\omega_2)\,\mathcal{L}_r(\omega_1\omega_2)\,\mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c'|}\right)S^*(\omega_1,\omega_2;c')\right|, \end{aligned} \tag{3.22}$$

with

$$\mathcal{N}(j) = 2^{-j/H} N \in [N/2, N] \qquad (j = 0, 1, \ldots, H), \tag{3.23}$$

$$\mathcal{L}_r(z) = \sum_{\substack{-L \le m \le L \\ m \equiv r \,(\mathrm{mod}\ 2)}} \eta'_m \left(\frac{z}{|z|}\right)^m \qquad (z \in \mathbb{C} - \{0\}) \tag{3.24}$$

and $a^{\pm}(\omega)$ and η'_m as in (3.20)-(3.21).

For later working it suffices that one takes, in the above,

$$H = \left[N^{\varepsilon/2}\right] \in \mathbb{N} \tag{3.25}$$

(and, of course, $h = h(H)$). Then, by applying the Cauchy-Schwarz inequality to bound the sum on the right-hand side of (3.22), one has (see (3.20), (3.7) and (1.9.16)):

$$\left|U^{\star}_{q_0,w,u}\left(\psi, c'; M_1, M_2; N\right)\right|^2 \le 2H^2 \mathcal{U}_H \left\|\mathbf{a}_N^{-}\right\|_2^2 = 2H^2 \mathcal{U}_H \left\|\mathbf{b}_N\right\|_2^2 \ll N^{\varepsilon} \mathcal{U}_H \left\|\mathbf{b}_N\right\|_2^2 \,, \tag{3.26}$$

where

$$\mathcal{U}_H = \sum_{r=0}^{1} \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \le N}} \left| \sum_{\substack{\omega' \in \mathfrak{O} \\ \mathcal{N}(h) < |\omega'|^2 \le \mathcal{N}(h-1)}} a^{+}(\omega')\, \mathcal{L}_r\left(\omega\omega'\right) \mathrm{e}\!\left(\frac{\psi\sqrt{|\omega\omega'|}}{|c'|}\right) S^{*}(\omega, \omega'; c') \right|^2 .$$

Choose now an infinitely differentiable function $G : \mathbb{R} \to [0,1]$ such that

$$G(x) = \begin{cases} 1 & \text{if } 1/2 \le x \le 1, \\ 0 & \text{if } x \le 1/4 \text{ or } x \ge 2, \end{cases}$$

and define g to be the complex function such that

$$g(z) = G\left(\frac{|z|^2}{N}\right) \qquad \text{for } z \in \mathbb{C}\,.$$

Given that G is chosen independently of all factors other than the above explicit requirements (as may, and shall, be assumed), one has $G^{(j)}(x) \ll_j 1$ for $j = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$. Moreover, since $g(\omega) = 1$ if $N/2 < |\omega|^2 \le N$, and is otherwise positive or zero, it follows that

$$\mathcal{U}_H \le \sum_{r=0}^{1} \sum_{\omega \in \mathfrak{O}} g(\omega) \left| \sum_{\substack{\omega' \in \mathfrak{O} \\ \mathcal{N}(h) < |\omega'|^2 \le \mathcal{N}(h-1)}} a^{+}(\omega')\, S^{*}(\omega, \omega'; c')\, \mathcal{L}_r\left(\omega\omega'\right) \mathrm{e}\!\left(\frac{\psi\sqrt{|\omega\omega'|}}{|c'|}\right) \right|^2 .$$

This, via recall of the definitions, (3.8) and (3.24), of $S^{*}(\omega, \omega'; c')$ and $\mathcal{L}_r(z)$, gives:

$$\begin{aligned} \mathcal{U}_H \le \quad & \sideset{}{^*}\sum_{\delta_1 \bmod c'\mathfrak{O}} \; \sideset{}{^*}\sum_{\delta_2 \bmod c'\mathfrak{O}} \; \sum_{\substack{-L \le m_1, m_2 \le L \\ m_1 \equiv m_2 \,(\mathrm{mod}\ 2)}} \eta'_{m_1} \overline{\eta'_{m_2}} \sum_{\substack{\omega'_1, \omega'_2 \in \mathfrak{O} \\ \mathcal{N}(h) < |\omega'_1|^2, |\omega'_2|^2 \le \mathcal{N}(h-1)}} a^{+}(\omega'_1)\, \overline{a^{+}(\omega'_2)} \;\times \\ & \times \left(\frac{\omega'_1}{|\omega'_1|}\right)^{m_1} \left(\frac{\omega'_2}{|\omega'_2|}\right)^{-m_2} \mathrm{e}\!\left(\mathrm{Re}\left(\frac{\delta_1\omega'_1 - \delta_2\omega'_2}{c'}\right)\right) W(\delta_1, \delta_2; m_1, m_2; \omega'_1, \omega'_2)\,, \end{aligned} \tag{3.27}$$

with the asterisks modifying the sums mod $c'\mathfrak{O}$ in the same way as in (2.12)-(2.13) of Lemma 2.3, while

$$W(\delta_1, \delta_2; m_1, m_2; \omega'_1, \omega'_2) = \sum_{\omega \in \mathfrak{O}} g(\omega) \left(\frac{\omega}{|\omega|}\right)^{m_1 - m_2} \mathrm{e}\!\left(\mathrm{Re}\!\left(\frac{(\alpha_1 - \alpha_2)}{c'}\,\omega\right) + \frac{\psi\left(\sqrt{|\omega'_1|} - \sqrt{|\omega'_2|}\right)}{|c'|}\,|\omega|^{1/2}\right)$$

where $\alpha_i \bmod c'\mathfrak{O}$ is the (unique) solution of the case $\delta \equiv \delta_i \bmod c'\mathfrak{O}$ of (2.14)-(2.15) of Lemma 2.3.

The summand of the last sum above may be expressed as $F\big(\mathrm{Re}(\omega), \mathrm{Im}(\omega)\big)$ where, since g was chosen so that $g(x+iy) = G\left(\left(x^2+y^2\right)/N\right)$ for $x, y \in \mathbb{R}$ (with the real function G having derivatives of all orders, and support that is a compact subset of $(0,\infty)$), one is able to deduce that the function $F : \mathbb{R}^2 \to \mathbb{C}$ lies in the Schwartz space. It therefore follows by (2.46)-(2.47) of Lemma 2.7 (Poisson's summation formula) that

$$W(\delta_1, \delta_2; m_1, m_2; \omega_1', \omega_2') = \sum_{\nu \in \mathfrak{O}} \hat{f}(\nu), \tag{3.28}$$

where

$$f(z) = g(z) \left(\frac{z}{|z|}\right)^{m_1 - m_2} \mathrm{e}\!\left(\frac{\psi\left(\sqrt{|\omega_1'|} - \sqrt{|\omega_2'|}\right)\sqrt{|z|}}{|c'|}\right) \mathrm{e}\!\left(\mathrm{Re}\!\left(\frac{(\alpha_1 - \alpha_2)}{c'}\, z\right)\right) \qquad (z \in \mathbb{C}),$$

so that in cases relevant to (3.27) one has (see (2.46)):

$$\hat{f}(\nu) = \hat{\varphi}\left(\nu + \frac{\alpha_2 - \alpha_1}{c'}\right) \qquad (\nu \in \mathfrak{O}) \tag{3.29}$$

with $\varphi : \mathbb{C} \to \mathbb{C}$ given by

$$\varphi(z) = g(z) \left(\frac{z}{|z|}\right)^{2d} \mathrm{e}\!\left(D\sqrt{|z|}\right) \qquad (z \in \mathbb{C}), \tag{3.30}$$

where

$$d = \frac{m_1 - m_2}{2} \in \mathbb{Z} \qquad \text{and} \qquad D = \frac{\psi\left(\sqrt{|\omega_1'|} - \sqrt{|\omega_2'|}\right)}{|c'|} \in \mathbb{R}\ . \tag{3.31}$$

Note here that, for the choices of m_1, m_2, ω_1' and ω_2' permitted in the sum on the right-hand side of (3.27), one has (using (3.18) and (1.9.29), (3.23) and (3.25)):

$$|d| \le L \le \frac{N^{(1-\varepsilon)/2}}{|c'|}\ . \tag{3.32}$$

and

$$|D| \le A_2\, \frac{(\mathcal{N}(h-1) - \mathcal{N}(h))}{4\,|c'|\,(N/2)^{3/4}} \asymp \frac{\left(2^{1/H} - 1\right) N^{1/4}}{|c'|} \asymp \frac{N^{(1/4)-(\varepsilon/2)}}{|c'|}\ . \tag{3.33}$$

By the result (2.49) of Lemma 2.8, estimates for $\hat{\varphi}(w)$ $(w \in \mathbb{C})$ follow from bounds on the functions $\mathbf{\Delta}_{\mathbb{C}}^{j}\varphi$ $(j = 0, 1, 2, \ldots)$, where $\mathbf{\Delta}_{\mathbb{C}}$ is Laplace's operator. Obtaining these bounds requires some preparation. Firstly, on recalling that $g(z) = G\left(|z|^2/N\right)$, one may rewrite (3.30) to obtain:

$$\varphi(z) = \Phi\big(|z|^2\big)\, z^d\, (\overline{z})^{-d} \qquad (z \in \mathbb{C} - \{0\}), \tag{3.34}$$

where

$$\Phi(x) = G(x/N)\, \mathrm{e}\!\left(Dx^{1/4}\right) \qquad (x > 0).$$

It is here moreover the case that, since $G^{(j)}(x) \ll_j 1$ and $(\partial/\partial x)^j\, \mathrm{e}\left(Dx^{1/4}\right) \ll_j \left(1 + Dx^{1/4}\right)^j x^{-j}$ for $x > 0$ and $j \in \mathbb{N} \cup \{0\}$, and since the support of G is contained in the interval $[1/4, 2]$, the formulae of Leibniz for the higher order derivatives of a product enable one to deduce that

$$x^j\, \Phi^{(j)}(x) \ll_j \left(1 + N^{1/4}|D|\right)^j \qquad (x > 0 \text{ and } j = 0, 1, 2, \ldots). \tag{3.35}$$

One may next use the decomposition

$$\mathbf{\Delta}_{\mathbb{C}} = 4\, \frac{\partial}{\partial z}\, \frac{\partial}{\partial \overline{z}}\ , \tag{3.36}$$

where the linear operators $\partial/\partial z$ and $\partial/\partial\overline{z}$ are defined as is indicated in (1.2.7). The decomposition (3.36) is useful for the matter in hand: for, when $q(z)$ is a complex function holomorphic on $\mathbb{C}-\{0\}$ (say), one has

$$\frac{\partial}{\partial z}\,q(z)=q'(z),\quad \frac{\partial}{\partial z}\,q\left(\overline{z}\right)=0,\quad \frac{\partial}{\partial\overline{z}}\,q(z)=0,\quad\text{and}\quad \frac{\partial}{\partial\overline{z}}\,q\left(\overline{z}\right)=q'\left(\overline{z}\right)\qquad (z\neq 0).$$

By the last observation, and the chain and product rules, one finds that

$$\frac{\partial}{\partial z}\,\Phi^{(j)}\left(|z|^2\right)=\overline{z}\,\Phi^{(j+1)}\left(|z|^2\right)\quad\text{and}\quad \frac{\partial}{\partial\overline{z}}\,\Phi^{(j)}\left(|z|^2\right)=z\Phi^{(j+1)}\left(|z|^2\right)\qquad (z\neq 0\text{ and } j=0,1,2,\dots).$$

By combining the facts just noted with Leibniz's formulae for derivatives of a product one may deduce from (3.34) and (3.36) that, for $0\neq z\in\mathbb{C}$ and $j=0,1,2,\dots$,

$$\begin{aligned}\boldsymbol{\Delta}_{\mathbb{C}}^{j}\varphi(z)&=\left(4\,\frac{\partial}{\partial z}\,\frac{\partial}{\partial\overline{z}}\right)^{j}\Phi\left(|z|^2\right)z^{d}\left(\overline{z}\right)^{-d}=\\ &=\left(4\,\frac{\partial}{\partial z}\right)^{j}\sum_{r=0}^{j}\binom{j}{r}z^{d+r}\,\Phi^{(r)}\big(|z|^2\big)\,(-d)(-d-1)\cdots(-d-(j-r-1))\,\left(\overline{z}\right)^{-d-(j-r)}=\\ &=4^{j}\sum_{r=0}^{j}\frac{j!}{r!}\binom{-d}{j-r}\left(\overline{z}\right)^{-d-(j-r)}\left(\frac{\partial}{\partial z}\right)^{j}z^{d+r}\,\Phi^{(r)}\big(|z|^2\big)\,,\end{aligned}$$

where $\binom{m}{n}$ is the coefficient of x^n in the binomial expansion of $(1+x)^m$. Similarly,

$$\left(\frac{\partial}{\partial z}\right)^{j}z^{d+r}\,\Phi^{(r)}\big(|z|^2\big)=\sum_{s=0}^{j}\frac{j!}{s!}\binom{d+r}{j-s}\left(\overline{z}\right)^{s}\Phi^{(r+s)}\left(|z|^2\right)z^{d+r-(j-s)}\,,$$

so that one obtains, for $z\neq 0$ and $j=0,1,2,\dots$,

$$\boldsymbol{\Delta}_{\mathbb{C}}^{j}\varphi(z)=\left(\frac{4}{|z|^2}\right)^{j}\left(\frac{z}{\overline{z}}\right)^{d}\sum_{r=0}^{j}\sum_{s=0}^{j}\frac{(j!)^2}{r!s!}\binom{-d}{j-r}\binom{d+r}{j-s}|z|^{2(r+s)}\Phi^{(r+s)}\left(|z|^2\right).$$

One may therefore deduce from the bounds in (3.35) that, for $0\neq z\in\mathbb{C}$ and $j=0,1,2,\dots$, one has

$$\boldsymbol{\Delta}_{\mathbb{C}}^{j}\varphi(z)=O_j\big(|z|^{-2j}\big)\sum_{r=0}^{j}\sum_{s=0}^{j}O_{j,r,s}\Big((1+|d|)^{(j-r)+(j-s)}\Big)\,O_{r+s}\Big(\Big(1+N^{1/4}|D|\Big)^{r+s}\Big)\ll_j\left(\frac{1+V}{|z|^2}\right)^{j},\quad (3.37)$$

where (given (3.32) and (3.33))

$$V=\max\Big\{d^2\,,\,N^{1/2}D^2\Big\}\ll\frac{N^{1-\varepsilon}}{|c'|^2}\,.\qquad (3.38)$$

Since one has here $N^{1-\varepsilon}/|c'|^2\gg 1$ (by (3.5) and (1.9.28)), and since $\varphi(z)=0$ unless $N/4\leq|z|^2\leq 2N$ (by (3.30) and the choice of g), one finds, by means of the result (2.49) of Lemma 2.8, that the bounds in (3.37) and (3.38) imply that one has

$$\hat{\varphi}(w)\ll_j N|w|^{-2j}\left(\frac{N^{1-\varepsilon}/|c'|^2}{N}\right)^{j}=N^{1-j\varepsilon}\,|c'w|^{-2j}\qquad (0\neq w\in\mathbb{C}\text{ and } j=0,1,2,\dots).\qquad (3.39)$$

The aim now is to apply (3.39) in order to estimate the sum $\sum_{\nu\in\mathfrak{O}}\hat{f}(\nu)$ on the right-hand side of (3.28). Let ν_0 be the unique Gaussian integer such that $\nu_0+(\alpha_2-\alpha_1)/c'\in\mathbb{Q}(i)$ has both its real and imaginary parts lying in the interval $(-1/2,1/2]$. Then, for $\nu\in\mathfrak{O}$ with $\nu\neq\nu_0$, one has

$$\left|\nu+\frac{\alpha_2-\alpha_1}{c'}\right|\geq|\nu-\nu_0|-\left|\nu_0+\frac{\alpha_2-\alpha_1}{c'}\right|\geq|\nu-\nu_0|-\frac{1}{\sqrt{2}}\gg|\nu-\nu_0|=|\nu'|\quad\text{(say)}.$$

From this, (3.29) and the case $j = 2 + [2/\varepsilon]$ of (3.39) it follows that

$$\sum_{\nu \in \mathfrak{O}} \hat{f}(\nu) = \hat{\varphi}\left(\nu_0 + \frac{\alpha_2 - \alpha_1}{c'}\right) + \sum_{0 \neq \nu' \in \mathfrak{O}} O_\varepsilon\left(N^{1-j\varepsilon} \left|c'\right|^{-2j} \left|\nu'\right|^{-2j}\right) =$$

$$= \hat{\varphi}\left(\nu_0 + \frac{\alpha_2 - \alpha_1}{c'}\right) + O_\varepsilon\left(N^{-1} \left|c'\right|^{-2j}\right), \tag{3.40}$$

where, by (3.3) and (3.5) (in which $vw \in \mathfrak{O}$ and $\epsilon \in \mathfrak{O}^*$), one has $0 \neq c' \in \mathfrak{O}$, so that $|c'| \geq 1$. If it is here the case that $\nu_0 + (\alpha_2 - \alpha_1)/c' \neq 0$, then one has

$$\left|\nu_0 + \frac{\alpha_2 - \alpha_1}{c'}\right| = \frac{|c'\nu_0 + \alpha_2 - \alpha_1|}{|c'|} \geq \frac{1}{|c'|},$$

so that, by (3.39),

$$\hat{\varphi}\left(\nu_0 + \frac{\alpha_2 - \alpha_1}{c'}\right) \ll_j N^{1-j\varepsilon} \qquad (j = 0, 1, 2, \dots).$$

Hence (and by (3.28) and (3.40)) one obtains:

$$W(\delta_1, \delta_2; m_1, m_2; \omega_1', \omega_2') = O_\varepsilon\left(N^{-1}\right) + \begin{cases} \hat{\varphi}(0) & \text{if } (\alpha_2 - \alpha_1)/c' \in \mathfrak{O}\,; \\ 0 & \text{otherwise.} \end{cases} \tag{3.41}$$

It should be noted here that $(\alpha_2 - \alpha_1)/c' \in \mathfrak{O}$ if and only if $\alpha_1 \equiv \alpha_2 \bmod c'\mathfrak{O}$. Moreover, for $i = 1, 2$, the relationship between $\alpha_i \bmod c'\mathfrak{O}$ and $\delta_i \bmod c'\mathfrak{O}$ is (as noted below (3.27)) the same as that existing between the variables $\alpha \bmod c'\mathfrak{O}$ and $\delta \bmod c'\mathfrak{O}$ in (2.13)-(2.15) of Lemma 2.3, so that by virtue of the final point noted in Lemma 2.3 one has $\alpha_1 \equiv \alpha_2 \bmod c'\mathfrak{O}$ if and only if $\delta_1 \equiv \delta_2 \bmod c'\mathfrak{O}$. In addition to this, it is easily seen that $\hat{\varphi}(0) = 0$ unless a further independent condition (on d) is satisfied: for, by (3.30), one has

$$\hat{\varphi}(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(z) \left(\frac{z}{|z|}\right)^{2d} \mathrm{e}\left(D\sqrt{|z|}\right) \mathrm{d}x \mathrm{d}y$$

(where z is a dependent variable satisfying $z = x + iy$), and so, given that $g(z) = g\left(|z|^2/N\right) = g(|z|)$ and that $d \in \mathbb{Z}$, a change to polar coordinates shows that

$$\hat{\varphi}(0) = \int_0^{2\pi} \int_0^{\infty} g(r) e^{2id\theta}\, \mathrm{e}\big(D\sqrt{r}\big)\, r\mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} e^{2id\theta} \mathrm{d}\theta \int_0^{\infty} g(r)\, \mathrm{e}\big(D\sqrt{r}\big)\, r\mathrm{d}r = E(2d) \int_0^{\infty} r g(r)\, \mathrm{e}\big(D\sqrt{r}\big)\, \mathrm{d}r\,,$$

where $E(n) = 2\pi$ if $n = 0$, while $E(n) = 0$ if $0 \neq n \in \mathbb{Z}$. Since the relevant values of d and D are those given by (3.31), it follows from (3.41) and the points just noted that one has:

$$W(\delta_1, \delta_2; m_1, m_2; \omega_1', \omega_2') = O_\varepsilon\left(N^{-1}\right) + \begin{cases} 2\pi \tilde{g}\left(\omega_1', \omega_2'\right) & \text{if } m_1 = m_2 \text{ and } \delta_1 \equiv \delta_2 \bmod c'\mathfrak{O}, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.42}$$

where

$$\tilde{g}\left(\omega_1', \omega_2'\right) = \int_0^{\infty} r g(r) \mathrm{e}\left(\frac{\psi\left(\sqrt{|\omega_1'|} - \sqrt{|\omega_2'|}\right)\sqrt{r}}{|c'|}\right) \mathrm{d}r =$$

$$= 2 \int_0^{(2N)^{1/4}} t^3 G\left(\frac{t^4}{N}\right) \mathrm{e}\left(\frac{\psi\left(\sqrt{|\omega_1'|} - \sqrt{|\omega_2'|}\right) t}{|c'|}\right) \mathrm{d}t\,. \tag{3.43}$$

By (3.27), (3.21) and (3.42), one obtains the upper bound

$$\mathcal{U}_H \le \mathcal{U}'_H + 2\pi \mathcal{U}''_H , \tag{3.44}$$

where, by (3.20), (3.23) and (3.7),

$$\begin{aligned} \mathcal{U}'_H &= \sideset{}{^*}\sum_{\delta_1 \bmod c'\mathfrak{O}} \sideset{}{^*}\sum_{\delta_2 \bmod c'\mathfrak{O}} \sum_{\substack{-L\le m_1,m_2\le L \\ m_1\equiv m_2 \pmod 2}} \sum_{\substack{\omega'_1,\omega'_2\in\mathfrak{O} \\ \mathcal{N}(h)<|\omega'_1|^2,|\omega'_2|^2\le\mathcal{N}(h-1)}} \left| a^+(\omega'_1)\, a^+(\omega'_2)\right| O_\varepsilon\left(N^{-1}\right) \ll \\ &\ll |c'|^4 (L+1)^2 \sum\sum_{\substack{\omega'_1,\omega'_2\in\mathfrak{O} \\ N/2<|\omega'_1|^2,|\omega'_2|^2\le N}} \left(|b(\omega'_1)|^2 + |b(\omega'_2)|^2\right) O_\varepsilon\left(N^{-1}\right) \ll_\varepsilon |c'|^4 (L+1)^2 \|\mathbf{b}_N\|_2^2 \end{aligned} \tag{3.45}$$

(with $\|\mathbf{b}_N\|_2$ as in (1.9.16)), while by (3.43) one has

$$\begin{aligned} \mathcal{U}''_H &= \sideset{}{^*}\sum_{\delta \bmod c'\mathfrak{O}} \sum_{m=-L}^{L} \sum\sum_{\substack{\omega'_1,\omega'_2\in\mathfrak{O} \\ \mathcal{N}(h)<|\omega'_1|^2,|\omega'_2|^2\le\mathcal{N}(h-1)}} a^+(\omega'_1)\left(\frac{\omega'_1}{|\omega'_1|}\right)^m \overline{a^+(\omega'_2)\left(\frac{\omega'_2}{|\omega'_2|}\right)^m} \mathrm{e}\left(\mathrm{Re}\left(\frac{\delta\left(\omega'_1-\omega'_2\right)}{c'}\right)\right) \tilde{g}\left(\omega'_1,\omega'_2\right) = \\ &= 2 \sideset{}{^*}\sum_{\delta \bmod c'\mathfrak{O}} \sum_{m=-L}^{L} \int_0^{(2N)^{1/4}} t^3 G\left(\frac{t^4}{N}\right) |s(\delta,m,t)|^2 \,\mathrm{d}t \ll \\ &\ll N^{3/4} \sideset{}{^*}\sum_{\delta \bmod c'\mathfrak{O}} \sum_{m=-L}^{L} \int_{-T}^{T} |s(\delta,m,t)|^2 \,\mathrm{d}t \, , \end{aligned} \tag{3.46}$$

with $T = (2N)^{1/4}$ and

$$s(\delta,m,t) = \sum_{\substack{\omega\in\mathfrak{O} \\ \mathcal{N}(h)<|\omega|^2\le\mathcal{N}(h-1)}} a^+(\omega)\left(\frac{\omega}{|\omega|}\right)^m \mathrm{e}\left(\mathrm{Re}\left(\frac{\delta\omega}{c'}\right) + \frac{\psi\sqrt{|\omega|}\; t}{|c'|}\right). \tag{3.47}$$

Given (3.23), and given that the hypothesis (1.9.29) implies $|\psi| > 0$, it follows by (3.46) and (3.47) that one may bound $\mathcal{U}''_H$ by applying the case $\alpha = \psi/|2c'|$, $\beta = -1/2$ of Lemma 2.6. By (3.23), (3.20) and (3.7), this application of Lemma 2.6 shows that

$$\begin{aligned} \mathcal{U}''_H &\ll N^{3/4}\left(|c'|\,(L+1) + N^{1/2}\right)\left(|c'|\,T + |\psi|^{-1}\,|c'|\,N^{1/4}\right)\|\mathbf{b}_N\|_2^2 \ll \\ &\ll \left(|c'|\,(L+1) + N^{1/2}\right)|\psi|^{-1}\,|c'|\,N\,\|\mathbf{b}_N\|_2^2 \end{aligned} \tag{3.48}$$

(the latter bound following since one has $N^{1/4} \asymp T$ and, by the hypothesis (1.9.29), $|\psi|^{-1} \gg 1$). By (3.5), the hypothesis (1.9.28) and (3.18), one has

$$|c'| \le |c'|\,(L+1) \ll N^{(1-\varepsilon)/2} \le N^{1/2} \, ,$$

so that from (3.44), (3.45) and (3.48) one may deduce:

$$\begin{aligned} \mathcal{U}_H &\le \left(O_\varepsilon\left(|c'|^4 (L+1)^2\right) + O\left(\left(|c'|\,(L+1) + N^{1/2}\right)|\psi|^{-1}\,|c'|\,N\right)\right)\|\mathbf{b}_N\|_2^2 \ll \\ &\ll \left(|\psi|^{-1} + O_\varepsilon\left(N^{-3\varepsilon/2}\right)\right)|c'|\,N^{3/2}\,\|\mathbf{b}_N\|_2^2 \, . \end{aligned}$$

By using this estimate for $\mathcal{U}_H$ in (3.26) and noting that $N^\varepsilon \ge 1$, one finds that

$$U^{\star}_{q_0,w,u}\left(\psi,c';M_1,M_2;N\right) \ll \left(|\psi|^{-1/2} + O_\varepsilon(1)\right)|c'|^{1/2}\,N^{(3+2\varepsilon)/4}\,\|\mathbf{b}_N\|_2^2 \, . \tag{3.49}$$

Finally, since $|c'| = |c|$ (by (3.5)), and since the definition (3.18) implies that

$$\frac{M+1}{L+1} \leq 1 + \frac{M}{L+1} \ll 1 + |c'|\, MN^{-(1-\varepsilon)/2} \,,$$

one finds that the bounds (3.19) and (3.49) combine to give (1.9.30) ■

§4. Further lemmas.

These lemmas are needed for the proof of Theorem 1, in the next section.

Lemma 4.1. *Let $0 \neq q_0 \in \mathfrak{O} = \mathbb{Z}[i]$ and $\tau \in \Gamma = \Gamma_0(q_0) \leq SL(2,\mathfrak{O})$. Suppose that $\mathfrak{a}$ and $\mathfrak{a}'$ are cusps of Γ such that $\tau\mathfrak{a} = \mathfrak{a}'$. Let $g_{\mathfrak{a}}, g_{\mathfrak{a}'} \in SL(2,\mathbb{C})$, with $g_{\mathfrak{a}'}$ such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} = \mathfrak{a}'$. Then (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} = \mathfrak{a}$ if and only if one has*

$$\tau g_{\mathfrak{a}} = g_{\mathfrak{a}'}\, h[\eta]\, n[\beta] \qquad \textit{for some } \beta, \eta \in \mathbb{C} \textit{ with } \eta^2 \in \mathfrak{O}^*. \tag{4.1}$$

Proof. The stated condition is necessary, since if (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} = \mathfrak{a}'$, and for $\mathfrak{c} = \mathfrak{a}$, then by the result (2.1) of Lemma 2.1 there must exist some $\eta \in \mathbb{C}$ with $\eta^2 \in \mathfrak{O}^*$ and some $\beta \in \mathbb{C}$ such that $g_{\mathfrak{a}'}^{-1}\tau g_{\mathfrak{a}} = h[\eta]\, n[\beta]$. The stated condition is also sufficient, since if (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} = \mathfrak{a}'$, then (4.1) implies both that

$$g_{\mathfrak{a}}\infty = \tau^{-1} g_{\mathfrak{a}'}\, h[\eta]\, n[\beta]\infty = \tau^{-1} g_{\mathfrak{a}'}\infty = \tau^{-1}\mathfrak{a}' = \mathfrak{a}$$

and that

$$\begin{aligned} g_{\mathfrak{a}}^{-1}\Gamma'_{\mathfrak{a}} g_{\mathfrak{a}} = g_{\mathfrak{a}}^{-1}\tau^{-1}\Gamma'_{\mathfrak{a}'}\tau g_{\mathfrak{a}} &= \left(\tau g_{\mathfrak{a}}\right)^{-1}\Gamma'_{\mathfrak{a}'}\tau g_{\mathfrak{a}} = \\ &= n[-\beta]\, h[1/\eta]\, g_{\mathfrak{a}'}^{-1}\,\Gamma'_{\mathfrak{a}'}\, g_{\mathfrak{a}'}\, h[\eta]\, n[\beta] = \\ &= n[-\beta]\, h[1/\eta]\, B^+\, h[\eta]\, n[\beta] = \\ &= \{n[-\beta]\, h[1/\eta]\, n[\alpha]\, h[\eta]\, n[\beta] : \alpha \in \mathfrak{O}\} = \\ &= \left\{n\left[\alpha/\eta^2\right] : \alpha \in \mathfrak{O}\right\} = \\ &= \{n\left[\alpha'\right] : \alpha' \in \mathfrak{O}\} = B^+ \quad \blacksquare \end{aligned}$$

Lemma 4.2. *Let $0 \neq q_0 \in \mathfrak{O} = \mathbb{Z}[i]$; let $\Gamma = \Gamma_0(q_0) \leq SL(2,\mathfrak{O})$; and let $B = B^+ \cup h[-1]B^+$, where B^+ is as in (1.1.21). Suppose also that $\mathfrak{a}$ is a cusp of Γ, and let $\mu(\mathfrak{a})$ be as described in Theorem 1. Then there exists $g_{\mathfrak{a}} \in SL(2,\mathbb{C})$ such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} = \mathfrak{a}$. Moreover, for each such $g_{\mathfrak{a}}$ one has either*

$$q_0\mu(\mathfrak{a}) \mid 2 \quad \textit{and} \quad g_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\, g_{\mathfrak{a}} = B \cup h[i]\, n[\beta_{\mathfrak{a}}]\, B \ \textit{ for some } \ \beta_{\mathfrak{a}} \in \mathbb{C}, \tag{4.2}$$

or else

$$q_0\mu(\mathfrak{a}) \nmid 2 \quad \textit{and} \quad g_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}} g_{\mathfrak{a}} = B\ . \tag{4.3}$$

Proof. Note firstly that, if $\mathfrak{a} = u/w$ with non-zero $u, w \in \mathfrak{O}$ such that $(u,w) \sim 1$ and $w \mid q_0$, then there exists a choice of $g_{\mathfrak{a}} \in \{g \in SL(2,\mathbb{C}) : g\infty = \mathfrak{a}\}$ such that one of the two statements (4.2), (4.3) is true. Indeed, supposing $u, w \in \mathfrak{O}$ to have the properties just listed, one may choose $\tilde{u}, \tilde{w} \in \mathfrak{O}$ such that

$$SL(2,\mathfrak{O}) \ni \begin{pmatrix} u & -\tilde{w} \\ w & \tilde{u} \end{pmatrix} = \tilde{g}_{u/w} \quad \text{(say)}. \tag{4.4}$$

Then $\tilde{g}_{u/w}\infty = u/w$. Moreover, since $SL(2,\mathfrak{O})$ is a group, it follows from (4.4) and (1.1.17)-(1.1.19) that

$$\begin{aligned} \tilde{g}_{u/w}^{-1}\Gamma_{u/w}\tilde{g}_{u/w} &= \tilde{g}_{u/w}^{-1}\Gamma\tilde{g}_{u/w} \cap P = \\ &= \left\{p \in P \cap SL(2,\mathfrak{O}) : \tilde{g}_{u/w}p\tilde{g}_{u/w}^{-1} \in \Gamma = \Gamma_0(q_0)\right\} = \\ &= \left\{\begin{pmatrix} \alpha & z \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathfrak{O}^*, z \in \mathfrak{O} \text{ and } \alpha\tilde{u}w - zw^2 - \alpha^{-1}\tilde{u}w \in q_0\mathfrak{O}\right\} = \\ &= \left\{\pm\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathfrak{O},\ wz \in (q_0/w)\mathfrak{O}\right\} \cup \left\{\pm\begin{pmatrix} i & z \\ 0 & -i \end{pmatrix} : z \in \mathfrak{O},\ wz - 2i\tilde{u} \in (q_0/w)\mathfrak{O}\right\}\ . \end{aligned}$$

Given that $(w,\tilde{u})\sim 1$, the congruence $wz\equiv 2i\tilde{u} \bmod (q_0/w)\mathfrak{O}$ is soluble if and only if $(w,q_0/w)\mid 2$. Hence, and by (1.9.15), it follows from the above that one has either

$$q_0\mu(u/w)\mid 2 \quad\text{and}\quad \tilde{g}_{u/w}^{-1}\Gamma_{\mathfrak{u}/w}\tilde{g}_{u/w}=\{\,h[\pm 1]\,n[q'\zeta]:\zeta\in\mathfrak{O}\}\cup\{h[\pm i]\,n[-iz_0+q'\zeta']:\zeta'\in\mathfrak{O}\}$$

or

$$q_0\mu(u/w)\not\mid 2 \quad\text{and}\quad \tilde{g}_{u/w}^{-1}\Gamma_{u/w}\tilde{g}_{u/w}=\{\,h[\pm 1]\,n[q'\zeta]:\zeta\in\mathfrak{O}\}\ ,$$

where (in either case) q' is any Gaussian integer such that $q'\sim(q_0/w)/(w,q_0/w)$, and where (in the former case) z_0 denotes an arbitrary Gaussian integer z_0 satisfying

$$\frac{w}{(w,q_0/w)}\,z_0\equiv\frac{2i\tilde{u}}{(w,q_0/w)} \bmod q'\mathfrak{O}\ .$$

Therefore, and since $h[1/u]h[\alpha]n[\beta]h[u]=h[\alpha]\,n\big[\beta/u^2\big]$, $h[\pm i]=h[\pm 1]\,h[i]$ and $n[\beta'+\zeta']=n[\beta']n[\zeta']$ (for $\alpha,u\in\mathbb{C}^*$ and $\beta,\beta',\zeta'\in\mathbb{C}$), one may ensure that one of the two statements (4.2), (4.3) holds for $\mathfrak{a}=u/w$ by putting $g_{u/w}=\tilde{g}_{u/w}\,h\big[\sqrt{q'}\big]$: note that for this choice of $g_{u/w}$ one also obtains the case $\mathfrak{c}=u/w$ of (1.1.16), since one has $g_{u/w}\infty=\tilde{g}_{u/w}\,h\big[\sqrt{q'}\big]\,\infty=\tilde{g}_{u/w}\infty=u/w$.

Since the claim made at the beginning of this proof has now been verified, it now only remains to be shown that what was claimed there does imply the lemma. In accordance with this aim, let it now be supposed that $\mathfrak{a}$ is some cusp of Γ. By the result (2.4) of Lemma 2.2, one has $\tau\mathfrak{a}=u/w$ for some $\tau\in\Gamma$, and some pair of non-zero Gaussian integers u,w such that $(u,w)\sim 1$ and $w\mid q_0$; moreover, by virtue of the claim verified in the first part of this proof, it follows that one may assign to any such pair u,w some $g_{u/w}\in SL(2,\mathbb{C})$ such that one has $g_{u/w}\infty=u/w$ and either

$$q_0\mu(u/w)\mid 2 \quad\text{and}\quad g_{u/w}^{-1}\Gamma_{u/w}g_{u/w}=B\cup h[i]\,n\big[\beta_{u/w}\big]\,B\ \text{ for some }\ \beta_{u/w}\in\mathbb{C}, \tag{4.5}$$

or

$$q_0\mu(u/w)\not\mid 2 \quad\text{and}\quad g_{u/w}^{-1}\Gamma_{u/w}g_{u/w}=B\ . \tag{4.6}$$

Since $B=B^+\cup h[-1]B^+$, where $B^+\le N\le P$ and N contains $n[\beta_{u/w}]$, but not $h[-1]$, $h[i]$ or $h[-i]$, it follows by (1.1.17) and (1.1.19) that in both the cases (4.5), (4.6) just mentioned one has

$$g_{u/w}^{-1}\Gamma'_{u/w}g_{u/w}=g_{u/w}^{-1}\Gamma_{u/w}\,g_{u/w}\cap N=B^+\cap N=B^+\ ,$$

so that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c}=u/w$. Therefore, and since one has $\tau\mathfrak{a}=u/w$ (with $\tau\in\Gamma$), it follows by Lemma 4.1 that the elements $g_\mathfrak{a}\in SL(2,\mathbb{C})$ such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c}=\mathfrak{a}$ are precisely those elements $g_\mathfrak{a}\in SL(2,\mathbb{C})$ which, for some $\beta,\eta\in\mathbb{C}$ with $\eta^2\in\mathfrak{O}^*$, satisfy:

$$\tau g_\mathfrak{a}=g_{u/w}\,h[\eta]\,n[\beta]\ . \tag{4.7}$$

Since $\mathfrak{O}^*\ni 1^2$, $\mathbb{C}\ni 0$ and $\tau^{-1}g_{u/w}\,h[1]n[0]=\tau^{-1}g_{u/w}\in SL(2,\mathbb{C})$, the observation just made completes the proof that there exists some $g_\mathfrak{a}\in SL(2,\mathbb{C})$ such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c}=\mathfrak{a}$.

For the final result of the lemma one may note that if (4.5) and (4.7) hold (with $\eta^2\in\mathfrak{O}^*$) then, since $\tau\mathfrak{a}=u/w$ (where $\tau\in\Gamma$), one will have

$$\begin{aligned} g_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\,g_\mathfrak{a}=g_\mathfrak{a}^{-1}\tau^{-1}\Gamma_{u/w}\,\tau g_\mathfrak{a}&=(\tau g_\mathfrak{a})^{-1}\,\Gamma_{u/w}\,\tau g_\mathfrak{a}=\\ &=n[-\beta]\,h[1/\eta]\,g_{u/w}^{-1}\Gamma_{u/w}\,g_{u/w}\,h[\eta]\,n[\beta]=\\ &=n[-\beta]\,h[1/\eta]\,\big(B\cup h[i]\,n\big[\beta_{u/w}\big]\,B\big)\,h[\eta]\,n[\beta]=\\ &=B\cup h[i]\,n\big[\eta^{-2}\beta_{u/w}+2\beta\big]\,B\ . \end{aligned} \tag{4.8}$$

If instead (4.6) and (4.7) hold (with $\beta,\eta\in\mathbb{C}$ and $\eta^2\in\mathfrak{O}^*$) then one will have (similarly):

$$g_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\,g_\mathfrak{a}=n[-\beta]\,h[1/\eta]\,Bh[\eta]\,n[\beta]=B\ .$$

In both the above two cases, the equation $\tau\mathfrak{a}=u/w$ implies the relation $\mu(\mathfrak{a})\sim\mu(u/w)$ ■

Remark 4.3. By (4.8) it is evident that when $q_0\mu(\mathfrak{a}) \mid 2$ one may choose $g_{\mathfrak{a}}$ such that (4.2) holds with $\beta_{\mathfrak{a}} = 0$. There will not, however, be any use made of this observation in what follows.

Lemma 4.4 (a Fourier transform). *For real y put*

$$G_n(y) = \int_{-\infty}^{\infty} x^n \exp\left(2ixy - x^2\right) \mathrm{d}x \qquad (n = 0, 1, 2, \dots\ ),$$

and set $G_{-1}(y)$ equal to zero. Then, when $y \in \mathbb{R}$, one has:

$$G_{2m}(y) = \int_{-\infty}^{\infty} x^{2m} e^{-x^2} \cos(2xy)\mathrm{d}x = 2\int_0^{\infty} x^{2m} e^{-x^2} \cos(2xy)\mathrm{d}x \qquad (m = 0, 1, 2, \dots\ ), \tag{4.9}$$

$$G_{2m+1}(y) = i\int_{-\infty}^{\infty} x^{2m+1} e^{-x^2} \sin(2xy)\mathrm{d}x = 2i\int_0^{\infty} x^{2m+1} e^{-x^2} \sin(2xy)\mathrm{d}x \qquad (m = 0, 1, 2, \dots\ ), \tag{4.10}$$

$$G_0(y) = \sqrt{\pi}\, e^{-y^2} \tag{4.11}$$

and

$$G_{n+1}(y) = iyG_n(y) + \frac{n}{2}\, G_{n-1}(y) \qquad (n = 0, 1, 2, \dots\ ). \tag{4.12}$$

Proof. Absolute convergence of the above integrals (for $n, m \geq 0$) may be established by means of the upper bound $x^k \exp\left(-x^2\right) \ll_k \exp(-|x|)$ (valid for $x \in \mathbb{R}$, when $k \geq 0$). The results (4.9) and (4.10) follow from Euler's identity (applied to $\exp(2ixy)$), coupled with the fact that if $f : \mathbb{R} \to \mathbb{C}$ is an odd integrable function then the integral $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ will, if it is absolutely convergent, be equal to zero. Indications as to a proof of (4.11) are given in Exercise 10.22 of [1]. The definition of $G_n(y)$ implies that $-2G_{n+1}(y) = \int_{-\infty}^{\infty} x^n \exp(2ixy)\, \mathrm{d}\left(\exp\left(-x^2\right)\right)$: integrating by parts, one obtains (4.12) ■

Lemma 4.5 (Bessel functions of integer order and the Neumannn-Graf addition formula). *Let $n \in \mathbb{Z}$, and take $J_n : \mathbb{C} \to \mathbb{C}$ to be the entire function which, for $z \neq 0$, satisfies $J_n(z) = (z/2)^n J_n^*(z)$, with $J_n^*(z)$ as defined in (1.9.6). Then, for all $z \in \mathbb{C}$, one has:*

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-ni\Theta + iz\sin(\Theta))\mathrm{d}\Theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\sin(\Phi) - n\Phi)\mathrm{d}\Phi\ , \tag{4.13}$$

$$|J_n(z)| \leq \min\left\{\exp\left(|\mathrm{Im}(z)|\right)\ ,\ \frac{|z/2|^{|n|}}{|n|!}\exp\left(|z|\right)\right\} \tag{4.14}$$

and, if $|\mathrm{Re}(z)| \gg n^2 + 1$, then

$$|J_n(z)| \ll \exp\left(|\mathrm{Im}(z)|\right)|\mathrm{Re}(z)|^{-1/2}\ . \tag{4.15}$$

If $n/2 = p \in \mathbb{Z}$, $0 \neq u \in \mathbb{C}$ and $e^{i\theta} = u/|u|$ then, for $y > 0$ such that $y^2 \neq e^{-2i\theta}$, one has

$$(-1)^p\left(\frac{ye^{i\theta} + (ye^{i\theta})^{-1}}{|ye^{i\theta} + (ye^{i\theta})^{-1}|}\right)^{2p} J_{2p}\left(|u|\left|ye^{i\theta} + (ye^{i\theta})^{-1}\right|\right) = \sum_{m=-\infty}^{\infty}(-1)^m J_{m+p}(y|u|)J_{m-p}\left(\frac{|u|}{y}\right)e^{2im\theta}\ , \tag{4.16}$$

and the sum on the right-hand side of this equation is continuous, as a function of y, on the interval $(0, \infty)$.

Proof. The first equality in (4.13) is a result established in Section 17.23 of [48]; the subsequent equality is an elementary deduction (utilising the identity $2\cos(\alpha) = e^{i\alpha} + e^{-i\alpha}$, a substitution $\Phi = -\phi$, and the periodicity of the relevant integrands). The first bound implicit in (4.14) is a trivial corollary of (4.13), for one has $|\exp(-ni\Theta + iz\sin(\Theta))| = \exp(-\mathrm{Im}(z)\sin(\Theta)) \leq \exp(|\mathrm{Im}(z)|)$ when Θ is real. Since (1.9.9), (1.9.8) and (1.9.6) imply the inequality $\left|J_n(z)\right| \leq |z/2|^{|n|}(|n|!)^{-1}J_0(i|z|)$, the second bound implicit in (4.14) follows from the first.

The upper bound (4.15) is obtained by applying the first derivative test, Lemma 5.1.2 of [18], in order to estimate the first integral appearing in (4.13): after partitioning the range of integration $[-\pi, \pi]$ into intervals within which the requisite monotonicity conditions are fulfilled, and having chosen some $\delta > 0$, one proceeds to refine the partition into a disjoint union of subintervals on which one has $|\cos(\Theta)| > (\delta + |n|)/|\mathrm{Re}(z)|$, and subintervals on which $|\cos(\Theta)| \leq (\delta + |n|)/|\mathrm{Re}(z)|$; the refined partition need consist of no more than eight subintervals; those on which $\cos(\Theta)$ is bounded away from zero will (by the first derivative test) contribute no more than $(4/\pi)\delta^{-1} \exp(|\mathrm{Im}(z)|)$ to the absolute value of the first integral in (4.13); the remaining subintervals contribute (by virtue of the implied bound on their length) no more than $(\delta + |n|)/|\mathrm{Re}(z)|$; hence one obtains the bound $\left|J_n(z)\right| \leq \left(4|\mathrm{Re}(\pi z)|^{-1/2} + |n||\mathrm{Re}(\pi z)|^{-1}\right) \exp(|\mathrm{Im}(z)|)$ by choosing $\delta = 2|\mathrm{Re}(z/\pi)|^{1/2}$; when $|\mathrm{Re}(z)| \gg n^2 + 1$ this gives (4.15).

The identity (4.16) is an application of Graf's generalisation, in Section 2 of [13], of Neumann's addition formula. Graf's result, as presented in Section 11.3 of [44], is that

$$J_\nu(\varpi) \left(\frac{Z - ze^{-i\phi}}{Z - ze^{i\phi}} \right)^{\nu/2} = \sum_{k=-\infty}^{\infty} J_{\nu+k}(Z) J_k(z) e^{ik\phi} , \tag{4.17}$$

where

$$\varpi = \sqrt{Z^2 + z^2 - 2Zz\cos(\phi)} = \sqrt{\left(Z - ze^{-i\phi}\right)\left(Z - ze^{i\phi}\right)} \tag{4.18}$$

and ν, Z, z and ϕ may take any complex values satisfying

$$|z/Z| < \exp(-|\mathrm{Im}(\phi)|) . \tag{4.19}$$

Suppose now that $n/2 = p \in \mathbb{Z}$. Then, according to a remark on Page 361 of [44], the case $\nu = 2p$ of (4.17)-(4.18) holds independently of (4.19). Since that remark is made without proof, it is worthwhile to verify it here. In doing so one is to assume that $\nu = 2p \in 2\mathbb{Z}$, and that Z and ϕ are given complex numbers with $Z \neq 0$. Then, by (1.9.9), (1.9.8) and (1.9.6), the complex function $w \mapsto J_\nu(w)$ is entire, even, and has a zero of order $|\nu| = 2|p|$ at $w = 0$. Consequently, given (4.18), the left-hand side of (4.17) is a function of z that is meromorphic on $\mathbb{C}$, and has no singularities other than (possibly) a removable singularity at one of the points $z = Ze^{\pm i\phi}$. Moreover, since $z \mapsto J_k(z)$ is an entire function when $k \in \mathbb{Z}$, and since the bound (4.14) shows the sum over k in (4.17) to be uniformly convergent for the values of z lying in any given compact subset of $\mathbb{C}$, it follows that the right-hand side of (4.17) is an entire function of the complex variable z. Since both sides of (4.17) are (apart from at most one removable singularity) entire functions of z, and since they are equal when z lies in the open disc where (4.19) is satisfied, it follows that (4.17) must hold for all $z \in \mathbb{C}$ (subject to suitable definition of the left-hand side at any removable singularity). This justifies the use of (4.17)-(4.18) whenever ν is an even integer, and Z, z and ϕ are complex numbers with $Z \neq 0$.

To deduce (4.16) one applies the identity (4.17)-(4.18) with $\nu = n = 2p$, $Z = |u|y$, $z = -|u|/y$ and $\phi = 2\theta$ (where, by hypothesis, $p \in \mathbb{Z}$, $0 \neq u \in \mathbb{C}$, $e^{i\theta} = u/|u|$ and $y > 0$). Then, in (4.17) and (4.18), one has:

$$\varpi^2 = |u|^2 \left(y^2 + y^{-2} + 2\cos(2\theta)\right) = |u|^2 \left| ye^{i\theta} + (ye^{i\theta})^{-1} \right|^2 ,$$

$$\left(\frac{Z - ze^{-i\phi}}{Z - ze^{i\phi}} \right)^{\nu/2} = \left(\frac{y + y^{-1}e^{-2i\theta}}{y + y^{-1}e^{2i\theta}} \right)^p = \left(\frac{ye^{i\theta} + y^{-1}e^{-i\theta}}{ye^{-i\theta} + y^{-1}e^{i\theta}} \right)^p e^{-2i\theta p}$$

and (using (1.9.9))

$$\begin{aligned} \sum_{k=-\infty}^{\infty} J_{\nu+k}(Z) J_k(z) e^{ik\phi} &= \sum_{k=-\infty}^{\infty} J_{k+2p}(y|u|) J_k(-|u|/y) e^{2im\theta} = \\ &= \sum_{m=-\infty}^{\infty} J_{m+p}(y|u|)(-1)^{m-p} J_{m-p}(|u|/y) e^{2i(m-p)\theta} ; \end{aligned}$$

since $J_{2p}(\varpi)$ is an even function of ϖ, and since $(\alpha/\overline{\alpha})^p = (\alpha/|\alpha|)^{2p}$, one therefore obtains the desired result (4.16) on multiplying by $(-1)^p e^{2ip\theta}$ both sides of the specified case of equation (4.17). Just as the right-hand

side of (4.17) was found to be entire, as a function of z, so it can also be established (by similar reasoning) that, for arbitrary fixed $p \in \mathbb{Z}$, $u \in \mathbb{C} - \{0\}$ and $\theta \in \mathbb{R}$, the right-hand side of (4.16) will be a function of y that is holomorphic on $\mathbb{C} - \{0\}$, and so will (in particular) be continuous for $y > 0$ ■

Lemma 4.6 (the B-transform of a certain test function). *Let $\sigma \in (1/2, 1)$ and $K, P \geq 1$; let h be the function on $\mathcal{S}_\sigma^\star = \{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{Z}$ given by*

$$h(\nu, p) = \exp\left((\nu/K)^2 - (p/P)^2\right) \quad \text{for } (\nu, p) \in \mathcal{S}_\sigma^\star . \tag{4.20}$$

Then there exist real numbers $\varrho, \vartheta > 3$ such that the conditions (i)-(iii) of Theorem B are satisfied.

Suppose moreover that $\mathbb{C} \ni u = |u|e^{i\theta} \neq 0$, that one has $1 \leq \Delta \in \mathbb{R}$, $M \in \mathbb{N}$ and

$$\Delta \leq \frac{M}{1 + |u|} \leq 2\Delta , \tag{4.21}$$

and that $\mathbf{B}h$ is the transform of h defined by (1.9.3)-(1.9.4) of Theorem B. Then

$$(\mathbf{B}h)(u) = \frac{1}{4\pi^3} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{P,K}(\eta, \xi) e^{-\xi^2 - \eta^2} A_M(\phi, \theta) \cos(|u| \psi(\xi/K, \eta/P; \phi)) \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}\phi + E_M(P, K; u) , \tag{4.22}$$

where

$$F_{P,K}(\eta, \xi) = \left(\frac{1}{2} - \eta^2\right) P^2 + \left(\frac{1}{2} - \xi^2\right) K^2 , \qquad \psi(y, x; \phi) = e^y \sin(\phi - x) + e^{-y} \sin(\phi + x) , \tag{4.23}$$

$$A_M(\phi, \theta) = \sum_{m=-M}^{M} (-1)^m \cos(2m\phi) e^{2im\theta} \tag{4.24}$$

and

$$E_M(P, K; u) \ll_j \left(P^2 + K^2\right)(1 + |u|)\Delta^{1-2j} \qquad (j \in \mathbb{N}). \tag{4.25}$$

At the same time, one has also

$$(\mathbf{B}h)(u) = \frac{|u|^2}{8\pi^3} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{P,K}(\eta, \xi) e^{-\xi^2 - \eta^2} A_M(\phi, \theta) \cos(|u| \psi(\xi/K, \eta/P; \phi)) \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}\phi + E_M(P, K; u) , \tag{4.26}$$

where

$$G_{P,K}(\eta, \xi) = \cosh(2\xi/K) - \cos(2\eta/P) = 2\left(\sinh^2(\xi/K) + \sin^2(\eta/P)\right) \tag{4.27}$$

(while $\psi(x, y; \phi)$, $A_M(\phi, \theta)$ and $E_M(P, K; u)$ remain as in (4.22)-(4.25)).

Proof. By (4.20) it is evident that $h(-\nu, p) = h(\nu, -p) = h(\nu, p)$, so that condition (i) of Theorem B is satisfied. Since the function $\nu \mapsto \exp\left(\nu^2/K^2\right)$ is entire, the definition (4.20) also ensures that $h(\nu, p)$ satisfies condition (ii) of Theorem B. One can moreover check that $h(\nu, p)$ satisfies condition (iii) of Theorem B for abitrary $\varrho, \vartheta > 3$ (a short calculation using the inequality $\exp\left(-x^2\right) \leq 1/\left(1 + x^4/2\right)$ gives, in particular, the case $\varrho = \vartheta = 4$ of the condition (iii), with an implicit constant not greater than $O\left(K^4 P^4\right) = O_h(1)$).

Suppose now that $0 \neq u \in \mathbb{C}$, and that $\theta \in \mathbb{R}$ satisfies $e^{i\theta} = u/|u|$. Since the definitions (1.9.4)-(1.9.6) of $\mathcal{K}_{\nu,p}(z)$ are equivalent to the equations (6.21) and (7.21) of [5], it follows by Bruggeman and Motohashi's identity (1.9.11) and the definition (1.9.3) of $(\mathbf{B}h)(z)$ that, for the chosen test function h (i.e. that in (4.20)), one has

$$(\mathbf{B}h)(u) = \frac{2}{\pi} \sum_{p \in \mathbb{Z}} \frac{(-1)^p}{e^{(p/P)^2}} \int_0^\infty \left(\frac{y e^{i\theta} + (y e^{i\theta})^{-1}}{|y e^{i\theta} + (y e^{i\theta})^{-1}|}\right)^{2p} J_{2p}\left(|u| \left|y e^{i\theta} + (y e^{i\theta})^{-1}\right|\right) f_p(y) \frac{\mathrm{d}y}{y} , \tag{4.28}$$

where, by an appeal to Lemma 4.4,

$$\begin{aligned}
f_p(y) = \frac{1}{4\pi i}\int\limits_{(0)} y^{2\nu} e^{(\nu/K)^2}\left(p^2-\nu^2\right)\mathrm{d}\nu &= \frac{K}{4\pi}\left(p^2 G_0(K\log y) + K^2 G_2(K\log y)\right) = \\
&= \frac{K}{4\pi}\left(p^2 + K^2\left(\frac{1}{2} - (K\log y)^2\right)\right) G_0(K\log y) = \\
&= \frac{K}{4\sqrt{\pi}}\left(p^2 + \frac{K^2}{2} - K^4\log^2 y\right) e^{-K^2\log^2 y}
\end{aligned} \tag{4.29}$$

(since $\left|y^{2\nu}\right| = 1$ for $y > 0$ and $\mathrm{Re}(\nu) = 0$, the change in the order of integration required to attain (4.28) is justified by the absolute convergence of the integrals in (4.29) and (1.9.11): see Theorem 10.40 of [1] and note that, by the bound (4.15) of Lemma 4.5, one has $J_{2p}\left(|u|\left|ye^{i\theta} + (ye^{i\theta})^{-1}\right|\right) \ll_{p,|u|} (y + y^{-1})^{-1/2}$ for $y > 0$).

By the result (4.16) of Lemma 4.5 (i.e. the Neumann-Graf addition law), the integrand in (4.28) is equal to

$$(-1)^p y^{-1} f_p(y) \sum_{m=-\infty}^{\infty} (-1)^m J_{m+p}(y|u|) J_{m-p}(y^{-1}|u|) e^{2im\theta} ,$$

and so, by substituting $-p$ for p, and $e^{\xi/K}$ for y, one may rewrite (4.28) as:

$$(\mathbf{B}h)(u) = \frac{2}{\pi K}\sum_{p\in\mathbb{Z}} e^{-(p/P)^2}\int\limits_{-\infty}^{\infty} f_p\left(e^{\xi/K}\right)\sum_{m\in\mathbb{Z}} J_{m-p}\left(e^{\xi/K}|u|\right) J_{m+p}\left(e^{-\xi/K}|u|\right)\left(\frac{iu}{|u|}\right)^{2m}\mathrm{d}\xi \tag{4.30}$$

($f_p(y)$ being, by (4.29), an even function of p). Then, by using the latter of the two integral representations of $J_n(z)$ appearing in the equations (4.13) of Lemma 4.5, one obtains (via some consideration of the periodicity of the relevant integrands) the result that

$$\begin{aligned}
&J_{m-p}\left(e^{\xi/K}|u|\right) J_{m+p}\left(e^{-\xi/K}|u|\right) = \\
&\quad = \frac{1}{8\pi^2}\iint\limits_{\substack{-2\pi\le\Psi-\Phi\le 2\pi\\ 0\le\Psi+\Phi\le 4\pi}} \cos\!\Big(e^{\xi/K}|u|\sin\Phi - (m-p)\Phi\Big)\cos\!\Big(e^{-\xi/K}|u|\sin\Psi - (m+p)\Psi\Big)\mathrm{d}\Phi\mathrm{d}\Psi .
\end{aligned}$$

On substituting $\delta = (\Psi - \Phi)/2$ and $\phi = (\Psi + \Phi)/2$, the integrand becomes a function of ϕ with period 2π. Hence one may take $[-\pi, \pi]$ as the range of integration for both δ and ϕ. Then, on using the identity $\cos(x)\cos(y) = \big(\cos(x + y) + \cos(x - y)\big)/2$ to express the integral as a sum of two integrals, one finds (by interchange of δ and ϕ, and the identity $\cos(-x) = \cos(x)$) that both integrals are equal. The outcome of these manipulations is that one obtains:

$$J_{m-p}\left(e^{\xi/K}|u|\right) J_{m+p}\left(e^{-\xi/K}|u|\right) = \frac{1}{4\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\cos(|u|\psi(\xi/K,\delta;\phi) - 2m\phi - 2p\delta)\mathrm{d}\delta\mathrm{d}\phi , \tag{4.31}$$

where $\psi(y, x; \phi)$ is as given by (4.23). The inner range of integration here may be changed to $[-\pi/2, 3\pi/2]$ (by periodicity of the integrand as a function of δ). Then, on writing the integral concerned as the sum of integrals over the subintervals $[-\pi/2, \pi/2]$ and $[\pi/2, 3\pi/2]$ (respectively), and substituting $\delta + \pi$ for δ in the latter integral, one finds that elementary trigonometric identities suffice to express the sum of the integrals over the two subintervals as the single integral

$$\int\limits_{-\pi/2}^{\pi/2} 2\cos(|u|\psi(\xi/K,\delta;\phi))\cos(2m\phi + 2p\delta)\mathrm{d}\delta .$$

Since $\cos(2m\phi+2p\delta)=\cos(2m\phi)\cos(2p\delta)-\sin(2m\phi)\sin(2p\delta)$, one may therefore reformulate (4.31) as:

$$J_{m-p}\left(e^{\xi/K}|u|\right)J_{m+p}\left(e^{-\xi/K}|u|\right)=\frac{1}{\pi}\int\limits_{-\pi/2}^{\pi/2}\left(\cos(2p\delta)c_{2m}(|u|,\xi/K,\delta)-\sin(2p\delta)s_{2m}(|u|,\xi/K,\delta)\right)\mathrm{d}\delta\ , \quad (4.32)$$

where

$$c_{2m}(r,y,x)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\cos(r\psi(y,x;\phi))\cos(2m\phi)\mathrm{d}\phi\ ,\qquad s_{2m}(r,y,x)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\cos(r\psi(y,x;\phi))\sin(2m\phi)\mathrm{d}\phi\ .$$

It is trivial that $|c_{2m}(r,y,x)|$ and $|s_{2m}(r,y,x)|$ are each bounded above by 1. Moreover, since the relevant functions have period 2π, two integrations by parts suffice in order to show that

$$c_{2m}(r,y,x)=-\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(\frac{\partial^2}{\partial\phi^2}\cos(r\psi(y,x;\phi))\right)\frac{\cos(2m\phi)}{4m^2}\,\mathrm{d}\phi\ ,$$

with a similar result holding in respect of $s_{2m}(r,y,x)$. By these results, along with the observations that, for $x,y,\phi\in\mathbb{R}$, the function ψ given by (4.23) is real-valued, and satisfies $\left|(\partial/\partial\phi)^j\,\psi(y,x;\phi)\right|\leq e^y+e^{-y}$ for $j=1,2$, one has:

$$|c_{2m}(r,y,x)|\ ,\ |s_{2m}(r,y,x)|\ll m^{-2}(1+r)r\cosh(2y)\qquad(0\neq m\in\mathbb{Z},\ x,y\in\mathbb{R},\ r>0).$$

Therefore, and by (4.32) and (4.29), the integral and innermost sum in (4.30) are uniformly convergent with respect to p. This justifies a change in the order of summation and integration, so that one may sum firstly over p, in (4.30), before going on to sum over m and integrate with respect to ξ. Terms involving the factor $\sin(2p\delta)$ (from (4.32)) cancel, since all other relevant factors are even functions of p. Hence (and by (4.29)) one obtains:

$$(\mathbf{B}h)(u)=\pi^{-3/2}\int\limits_{-\infty}^{\infty}e^{-\xi^2}\sum_{m\in\mathbb{Z}}\left(\frac{iu}{|u|}\right)^{2m}T_m(|u|,\xi;P,K)\mathrm{d}\xi\ , \quad (4.33)$$

where

$$T_m(r,\xi;P,K)=\frac{1}{2\pi}\int\limits_{-\pi/2}^{\pi/2}c_{2m}(r,\xi/K,\delta)\left(\sum_{p\in\mathbb{Z}}\cos(2p\delta)\left(p^2+\left(\frac{1}{2}-\xi^2\right)K^2\right)e^{-(p/P)^2}\right)\mathrm{d}\delta. \quad (4.34)$$

One now repeats a calculation from the proof of Corollary 10.1 of [5], by applying Poisson summation (i.e. the case $n=1$ of the equation (2.45) of Lemma 2.7) to the sum over p in the last equation; then, after evaluating the relevant Fourier integrals (very similar to the integral appearing in (4.29)) by means of Lemma 4.4, one finds that the sum over p in (4.34) equals

$$\sqrt{\pi}P\sum_{v\in\mathbb{Z}}\left(\left(\frac{1}{2}-P^2(\pi v+\delta)^2\right)P^2+\left(\frac{1}{2}-\xi^2\right)K^2\right)e^{-P^2(\pi v+\delta)^2}\ .$$

Therefore, and given that $c_{2m}(r,\xi/K,\delta)=c_{2m}(r,\xi/K,v\pi+\delta)$ for $v\in\mathbb{Z}$ (as follows by (4.23) and the definition of $c_{2m}(r,y,x)$), one finds that

$$T_m(r,\xi;P,K)=\frac{P}{2\sqrt{\pi}}\sum_{v\in\mathbb{Z}}\int\limits_{-\pi/2}^{\pi/2}g(\pi v+\delta)\mathrm{d}\delta=\frac{P}{2\sqrt{\pi}}\int\limits_{-\infty}^{\infty}g(x)\mathrm{d}x,$$

where

$$g(x) = \left(\left(\frac{1}{2} - P^2 x^2 \right) P^2 + \left(\frac{1}{2} - \xi^2 \right) K^2 \right) e^{-P^2 x^2} c_{2m}(r, \xi/K, x) \ .$$

Recalling now the definition of $c_{2m}(r,y,x)$, and substituting η/P for x (in the above), one obtains:

$$T_m(r,\xi;P,K) = \frac{1}{4\pi^{3/2}} \int_{-\infty}^{\infty} F_{P,K}(\eta,\xi) e^{-\eta^2} \int_{-\pi}^{\pi} \cos(r\psi(\xi/K, \eta/P;\phi)) \cos(2m\phi) \mathrm{d}\phi \mathrm{d}\eta \ , \tag{4.35}$$

where $F_{P,K}(\eta,\xi)$ is as given by (4.23).

For $j \in \mathbb{N}$, one can show (using $2j$ integrations-by-parts) that

$$\int_{-\pi}^{\pi} \cos(r\psi(y,x;\phi)) \cos(2m\phi) \mathrm{d}\phi \ll_j m^{-2j} \left(1 + r^{2j} \cosh(2jy)\right) \qquad (0 \neq m \in \mathbb{Z},\ x,y \in \mathbb{R}).$$

By these bounds, and (4.35) and (4.23), it follows that if $j \in \mathbb{N}$ then

$$T_m(r,\xi;P,K) \ll_j m^{-2j} \left(P^2 + \left(1+\xi^2\right) K^2\right) \left(1 + r^{2j} \cosh(2j\xi/K)\right) \qquad (0 \neq m \in \mathbb{Z},\ \xi \in \mathbb{R},\ r > 0),$$

so that, since $K \geq 1$, one obtains:

$$\sum_{\substack{m \in \mathbb{Z} \\ |m| > M}} |T_m(r,\xi;P,K)| \ll_j (M+1)^{1-2j} \left(P^2 + \left(1+\xi^2\right) K^2\right) \left(1 + r^{2j} \cosh(2j\xi)\right) \qquad (M \geq 0,\ \xi \in \mathbb{R},\ r > 0).$$

Suppose now that $\Delta \geq 1$, and that $M \in \mathbb{N}$ satisfies the conditions in (4.21). Then it follows, by (4.33) and the last bound obtained, that for $j \in \mathbb{N}$ one has

$$(\mathbf{B}h)(u) - \frac{1}{\pi^{3/2}} \int_{-\infty}^{\infty} e^{-\xi^2} \sum_{m=-M}^{M} \left(\frac{iu}{|u|} \right)^{2m} T_m(|u|,\xi;P,K) \mathrm{d}\xi = O_j \left(M^{1-2j} \left(P^2 + K^2\right) \left(1 + |u|^{2j}\right) \right) \ll_j$$

$$\ll_j \left(P^2 + K^2\right) (1+|u|) \Delta^{1-2j} \ .$$

This, together with (4.35) (and justifiable changes in the order of summation and integration), yields the first main estimate of the lemma, as expressed in (4.22)-(4.25).

It now only remains to show how (4.26)-(4.27) follow. Let $I(P',K')$ denote the triple integral got by substituting $F_{P',K'}(\eta,\xi)$ in place of the factor $F_{P,K}(\eta,\xi)$ in the integrand in equation (4.22). Then, by (4.23), the integral in (4.22) may be expressed as the sum $I(P,0) + I(0,K)$. To transform $I(P,0)$ one integrates by parts twice (with respect to η), using:

$$\int (1/2 - \eta^2) e^{-\eta^2} \mathrm{d}\eta = (1/2)\eta e^{-\eta^2} + C_1 \quad \text{and} \quad \int \eta e^{-\eta^2} \mathrm{d}\eta = -(1/2) e^{-\eta^2} + C_2 \ .$$

After transforming $I(0,K)$ similarly (through integrations by parts with respect to ξ), one observes a certain obvious cancellation between terms of the expression obtained for $I(P,0)$ and terms of the expression obtained for $I(0,K)$: it then requires only a few more steps (involving use of elementary trigonometric identities) to arrive at the result expressed in (4.26)-(4.27) ■

Remark 4.7. The definition (4.24) expresses the function $A_M(\phi,\theta)$ in the form most convenient for use in this paper. It may nevertheless be worth noting that, by (4.24), one in fact has:

$$A_M(\phi,\theta) = \sum_{m=-M}^{M} (-1)^m \cos(2m\phi) \cos(2m\theta) = \frac{(-1)^M}{2} \left(\frac{\cos((2M+1)(\phi+\theta))}{\cos(\phi+\theta)} + \frac{\cos((2M+1)(\phi-\theta))}{\cos(\phi-\theta)} \right),$$

for $M \in \mathbb{N} \cup \{0\}$ and $\phi, theta \in \mathbb{R}$ such that $\cos(\phi+\theta)\cos(\phi-\theta) \neq 0$.

Lemma 4.8. *Let $-\pi/2 < x < \pi/2$; and let $y \in \mathbb{R}$. Suppose that*

$$J = \int_{-\pi}^{\pi} |\psi(y,x;\phi)|^{-1/2}\, \mathrm{d}\phi \, ,$$

where $\psi(y,x;\phi)$ is as defined in (4.23). Then $J \ll (\cos(x))^{-1/2}$.

Proof. Using $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$, one finds that

$$\psi(y,x;\phi) = 2\left(\cosh(y)\cos(x)\sin(\phi) - \sinh(y)\sin(x)\cos(\phi)\right) = 2\left(A\cos(\phi) + B\sin(\phi)\right),$$

where $A = -\sinh(y)\sin(x)$ and $B = \cosh(y)\cos(x)$. Therefore, on putting

$$Z = Z(x,y) = \sqrt{A^2+B^2} = \sqrt{\cosh^2(y) - \sin^2(x)},$$

one has $A/Z = \cos(\alpha)$ and $B/Z = \sin(\alpha)$ for some $\alpha = \alpha(x,y)$ (obviously independent of ϕ), and so

$$\psi(y,x;\phi) = 2Z(x,y)\cos\left(\phi - \alpha(x,y)\right) \qquad (\phi \in \mathbb{R}).$$

Using this, one obtains:

$$J \ll Z(x,y)^{-1/2} \int_{-\pi}^{\pi} \left|\cos\left(\phi - \alpha(x,y)\right)\right|^{-1/2} \mathrm{d}\phi = 4Z(x,y)^{-1/2} \int_{0}^{\pi/2} (\sin(\theta))^{-1/2}\, \mathrm{d}\theta \ll Z(x,y)^{-1/2} \, .$$

The lemma follows, since $Z^2 = \cosh^2(y) - \sin^2(x) \geq 1 - \sin^2(x) = \cos^2(x)$ ■

§5. The proof of Theorem 1.

Let the hypotheses of the theorem hold; let $\mathfrak{a}$ be a cusp of Γ with $\mathfrak{a} \stackrel{\Gamma}{\sim} u/w$; and let $E_j^{\mathfrak{a}}(q_0,P,K;N,b)$ $(j=0,1)$ and $\mu(\mathfrak{a})$ be as given by (1.9.12), (1.9.13) and (1.9.15). Suppose also that $1/2 < \sigma < 1$, and that

$$h(\nu,p) = \exp\left(\left(\frac{\nu}{K}\right)^2 - \left(\frac{p}{P}\right)^2\right) \qquad \text{for } p \in \mathbb{Z} \text{ and } \nu \in \mathbb{C} \text{ with } |\mathrm{Re}(\nu)| \leq \sigma \tag{5.1}$$

(so that $h(\nu,p)$ is the function $h : \mathcal{S}_\sigma^\star \to \mathbb{C}$ defined in (4.20) of Lemma 4.6). Then $h(\nu,p) \geq \exp(-2)$ for all pairs $(\nu,p) \in ((i\mathbb{R}) \cup [-\sigma,\sigma]) \times \mathbb{Z}$ such that $|\nu| \leq K$ and $|p| \leq P$. Moreover, each irreducible subspace $V \subset {}^0L^2(\Gamma\backslash G)$ that is the index of a summand on the right-hand side of Equation (1.9.12) is also the index of a summand on the left-hand side of Equation (1.9.1), and so it may be inferred from Remark 1.9.2, below Theorem B, that each pair of spectral parameters ν_V, p_V associated with a summand on the right-hand side of Equation (1.9.12) is such that the condition (1.9.7) is satisfied. Therefore, with $h(\nu,p)$ as in (5.1) (where, by hypothesis, $\sigma > 1/2 > 2/9$), it follows from the definitions (1.9.12) and (1.9.13) that one has

$$0 \leq E_j^{\mathfrak{a}}(q_0,P,K;N,b) \leq \sum_{k=0}^{1} E_k^{\mathfrak{a}}(q_0,P,K;N,b) \leq e^2 \mathcal{E}^{\mathfrak{a}}(q_0,P,K;N,b) \qquad \text{for } j = 0,1\, , \tag{5.2}$$

where

$$\begin{aligned}\mathcal{E}^{\mathfrak{a}}(q_0,P,K;N,b) = \sum_V \Bigg| \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} b(\omega) C_V^{\mathfrak{a}}(\omega;\nu_V,p_V) \Bigg|^2 \exp\left(\frac{\nu_V^2}{K^2} - \frac{p_V^2}{P^2}\right) + \\ + \sum_{\mathfrak{c} \in \mathfrak{C}(\Gamma)} \frac{1}{4\pi i\, [\Gamma_{\mathfrak{c}} : \Gamma_{\mathfrak{c}}']} \sum_{p \in \frac{1}{2}[\Gamma_{\mathfrak{c}}:\Gamma_{\mathfrak{c}}']\mathbb{Z}} \int_{(0)} \Bigg| \sum_{\substack{\omega \in \mathfrak{O} \\ N/2 < |\omega|^2 \leq N}} b(\omega) B_{\mathfrak{c}}^{\mathfrak{a}}(\omega;\nu,p) \Bigg|^2 \exp\left(\frac{\nu^2}{K^2} - \frac{p^2}{P^2}\right) \mathrm{d}\nu \, .\end{aligned} \tag{5.3}$$

After using the identity $|S|^2 = \overline{S}\,S$ to expand the squared absolute values in (5.3), one may change the order of summation so as to rewrite the right-hand side of equation (5.3) in the form

$$\sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\leq N}}\sum \overline{b\left(\omega_1\right)}\, b\left(\omega_2\right) H_{\mathfrak{a},\mathfrak{a}}\left(\omega_1,\omega_2\right),$$

where $H_{\mathfrak{a},\mathfrak{b}}\left(\omega_1,\omega_2\right)$ denotes the left-hand side of the special case of equation (1.9.1) in which h is the function defined in (5.1). By Lemma 4.6 it follows that, when σ and the function h are as we suppose in (5.1), there exists some pair of real numbers $\varrho,\vartheta > 3$ such that the conditions (i)-(iii) of Theorem B are satisfied (the proof of Lemma 4.6 shows, specifically, that one may take $\varrho = \vartheta = 4$ here). Hence one may apply the spectral sum formula (Theorem B) to each term $H_{\mathfrak{a},\mathfrak{a}}\left(\omega_1,\omega_2\right)$ in the above sum, and so obtain:

$$\mathcal{E}^{\mathfrak{a}}(q_0,P,K;N,b) = \mathcal{D}^{\mathfrak{a}}(q_0,P,K;N,b) + \mathcal{L}^{\mathfrak{a}}(q_0,P,K;N,b)\ , \tag{5.4}$$

where

$$\mathcal{D}^{\mathfrak{a}}(q_0,P,K;N,b) = \frac{1}{4\pi^3 i}\left(\sum_{p\in\mathbb{Z}}\int_{(0)} h(\nu,p)\left(p^2-\nu^2\right)\mathrm{d}\nu\right)\sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\leq N}}\sum \delta^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_2}\,\overline{b\left(\omega_1\right)}\, b\left(\omega_2\right) \tag{5.5}$$

and

$$\mathcal{L}^{\mathfrak{a}}(q_0,P,K;N,b) = \sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\leq N}}\sum \overline{b\left(\omega_1\right)}\, b\left(\omega_2\right)\sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}}\frac{S_{\mathfrak{a},\mathfrak{a}}\left(\omega_1,\omega_2;c\right)}{|c|^2}\left(\mathbf{B}h\right)\!\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c}\right), \tag{5.6}$$

with h as in (5.1), and with $\delta^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_2}$ and $(\mathbf{B}h)(z)$ as given by (1.9.2) and (1.9.3)-(1.9.6).

By (1.9.2) one has, in equation (5.5),

$$\delta^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_2} = \sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma_{\mathfrak{a}}\\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{a}}=\begin{pmatrix}u(\gamma) & \beta(\gamma)\\ 0 & 1/u(\gamma)\end{pmatrix}}} \mathrm{e}\left(\mathrm{Re}\left(\beta(\gamma)u(\gamma)\omega_1\right)\right)\delta_{u(\gamma)\omega_1,\omega_2/u(\gamma)}\ .$$

For a more explicit representation of the last sum note firstly that the scaling matrix $g_{\mathfrak{a}}$ is, by hypothesis, such that that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c}=\mathfrak{a}$. Therefore $\Gamma'_{\mathfrak{a}} = g_{\mathfrak{a}}B^+g_{\mathfrak{a}}^{-1}$, where $B^+ = \{n[\alpha] : \alpha\in\mathfrak{O}\}$. This, when combined with the results of Lemma 4.2 concerning $g_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}g_{\mathfrak{a}}$, shows that there exists a complex number $\beta_{\mathfrak{a}}$ such that the set $\mathcal{T}_{\mathfrak{a}}$ given by

$$\mathcal{T}_{\mathfrak{a}} = \begin{cases} g_{\mathfrak{a}}\big\{h[1],h[-1],h[i]\,n[\beta_{\mathfrak{a}}]\,,h[-i]\,n[\beta_{\mathfrak{a}}]\big\}g_{\mathfrak{a}}^{-1} & \text{if } q_0\mu(\mathfrak{a})\mid 2,\\ g_{\mathfrak{a}}\left\{h[1],h[-1]\right\}g_{\mathfrak{a}}^{-1} & \text{otherwise,}\end{cases} \tag{5.7}$$

is a complete set of coset representatives of $\Gamma'_{\mathfrak{a}}$ in $\Gamma_{\mathfrak{a}}$: as noted below (1.1.19), the group $\Gamma'_{\mathfrak{a}}$ is a normal subgroup of $\Gamma_{\mathfrak{a}}$ (one has, in particular, $h[i]\,n[\beta_{\mathfrak{a}}]\,B^+ = B^+h[i]\,n[\beta_{\mathfrak{a}}]$). Consequently one finds that if $q_0\mu(\mathfrak{a})\mid 2$ then

$$\delta^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_2} = 2\delta_{\omega_1,\omega_2} + 2\mathrm{e}\left(\mathrm{Re}\left(i^2\beta_{\mathfrak{a}}\omega_1\right)\right)\delta_{i\omega_1,\omega_2/i} = 2\big(\delta_{\omega_1,\omega_2} + \mathrm{e}\left(-\mathrm{Re}\left(\beta_{\mathfrak{a}}\omega_1\right)\right)\delta_{-\omega_1,\omega_2}\big)\ ,$$

while if instead $q_0\mu(\mathfrak{a})\not\mid 2$ then one has simply $\delta^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_2} = 2\delta_{\omega_1,\omega_2}$. Therefore, and since (as (5.7) shows)

$$[\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}] = \begin{cases}4 & \text{if } q_0\mu(\mathfrak{a})\mid 2\,,\\ 2 & \text{otherwise,}\end{cases} \tag{5.8}$$

it follows that one has:

$$\sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\leq N}}\sum \delta^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_2}\,\overline{b\left(\omega_1\right)}\, b\left(\omega_2\right) = [\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]\sum_{\substack{\omega\in\mathfrak{O}\\ N/2<|\omega|^2\leq N}}\left|b^{\mathfrak{a}}(\omega)\right|^2 = [\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]\ \|\mathbf{b}^{\mathfrak{a}}_N\|_2^2\ , \tag{5.9}$$

where, for $0 \neq \omega \in \mathfrak{O}$,

$$b^{\mathfrak{a}}(\omega) = \begin{cases} \frac{1}{2}\, b(\omega) \mathrm{e}\left(\mathrm{Re}\left(\frac{1}{2}\beta_{\mathfrak{a}}\omega\right)\right) + \frac{1}{2}\, b(-\omega) \mathrm{e}\left(-\mathrm{Re}\left(\frac{1}{2}\beta_{\mathfrak{a}}\omega\right)\right) & \text{if } q_0\mu(\mathfrak{a}) \mid 2\,; \\ b(\omega) & \text{otherwise.} \end{cases} \tag{5.10}$$

Regarding the other factors on the right-hand side of (5.5), one finds that, for h as in (5.1) and $p \in \mathbb{Z}$,

$$\begin{aligned} \frac{1}{4\pi^3 i} \int_{(0)} h(\nu, p) \left(p^2 - \nu^2\right) \mathrm{d}\nu &= \frac{1}{4\pi^3} \exp\left(-(p/P)^2\right) \int_{-\infty}^{\infty} \left(p^2 + t^2\right) \exp\left(-(t/K)^2\right) \mathrm{d}t = \\ &= \frac{K}{4\pi^3} \exp\left(-(p/P)^2\right) \int_{-\infty}^{\infty} \left(p^2 + K^2 x^2\right) \exp\left(-x^2\right) \mathrm{d}x = \\ &= \frac{K}{4\pi^3} \exp\left(-(p/P)^2\right) \left(p^2 G_0(0) + K^2 G_2(0)\right), \end{aligned} \tag{5.11}$$

where $G_0(0)$ and $G_2(0)$ are the constants given by (4.11)-(4.12) of Lemma 4.4. Moreover, by Poisson sumation over $\mathbb{Z}$ (i.e. the case $n = 1$ of results (2.44)-(2.45) of Lemma 2.7) , one finds that if k is a non-negative integer then

$$\sum_{p=-\infty}^{\infty} p^{2k} \exp\left(-(p/P)^2\right) = \sum_{v=-\infty}^{\infty} \int_{-\infty}^{\infty} y^{2k} \exp\left(-(y/P)^2\right) \mathrm{e}(-vy) \mathrm{d}y = P^{1+2k} \sum_{v=-\infty}^{\infty} G_{2k}(-\pi P v)\,, \tag{5.12}$$

where $G_0(Y), G_1(Y), G_2(Y), \ldots$ are given by (4.11)-(4.12) of Lemma 4.4, so that one has (in particular):

$$G_2(Y) = iYG_1(Y) + \frac{1}{2}\, G_0(Y) = \left(\frac{1}{2} - Y^2\right) G_0(Y) = \left(\frac{1}{2} - Y^2\right) \sqrt{\pi}\, \exp\left(-Y^2\right) \qquad (Y \in \mathbb{R}). \tag{5.13}$$

By (5.11), (5.12) (for $k = 0, 2$) and (5.13), one obtains

$$\begin{aligned} \frac{1}{4\pi^3 i} \sum_{p=-\infty}^{\infty} \int_{(0)} h(\nu, p) \left(p^2 - \nu^2\right) \mathrm{d}\nu &= \frac{KP}{4\pi^2} \sum_{v=-\infty}^{\infty} \left(P^2 \left(\frac{1}{2} - (\pi P v)^2\right) + \frac{K^2}{2}\right) \exp\left(-(\pi P v)^2\right) = \\ &= \frac{1}{8\pi^2}\, KP\left(K^2 + P^2\right) \left(1 + O\left(P^2 e^{-\pi^2 P^2}\right)\right) \quad \text{for } K, P \geq 1\,. \end{aligned} \tag{5.14}$$

By applying the results of (5.9) and (5.14) one may now conclude that

$$\mathcal{D}^{\mathfrak{a}}(q_0, P, K; N, b) = \frac{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]}{8\pi^2} \left(PK^3 + P^3 K\right) \left(1 + O\left(P^2 e^{-\pi^2 P^2}\right)\right) \|\mathbf{b}^{\mathfrak{a}}_N\|_2^2\ , \tag{5.15}$$

where the function $b^{\mathfrak{a}} : \mathfrak{O} - \{0\} \rightarrow \mathbb{C}$ is given by (5.10), and the meaning of $\|\mathbf{b}^{\mathfrak{a}}_N\|_2$ is consistent with (1.9.16).

In order to complete this proof it is necessary to obtain a suitable upper bound for the absolute value of the term $\mathcal{L}^{\mathfrak{a}}(q_0, P, K; N, b)$ in (5.4). As a first step towards this one may observe that, by (5.6), one has:

$$\mathcal{L}^{\mathfrak{a}}(q_0, P, K; N, b) = \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}} \frac{\Phi(c; P, K; N)}{|c|^2}\ , \tag{5.16}$$

where, for $0 \neq c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}$,

$$\Phi(c; P, K; N) = \sum_{\substack{\omega_1, \omega_2 \in \mathfrak{O} \\ N/2 < |\omega_1|^2, |\omega_2|^2 \leq N}} \sum (\mathbf{B}h)\!\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c}\right) S_{\mathfrak{a},\mathfrak{a}}(\omega_1, \omega_2; c)\, \overline{b\,(\omega_1)}\, b\,(\omega_2)\,. \tag{5.17}$$

No attempt will be made to extract a saving from the averaging over c that is apparent in the above: the approach taken will instead be to bound, individually, the absolute value of each term in the sum on the right-hand side of (5.16). Therefore suppose now that $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}$. Then, as is recorded in Proposition 2, the number c satisfies (1.9.24), and is consequently (see (1.9.15), and Remark 1.9.7 below it) a non-zero Gaussian integer. Moreover, given Lemma 4.2, it follows trivially from the results (2.3), (2.4) and (2.12) of Lemma 2.1, Lemma 2.2 and Lemma 2.3 that

$$|S_{\mathfrak{a},\mathfrak{a}}(\omega_1,\omega_2;c)| \le \sum_{\delta \bmod c\mathfrak{O}} 1 = |c|^2 \qquad (\omega_1,\omega_2 \in \mathfrak{O}). \tag{5.18}$$

Given (5.1), one may prepare for the estimation of $\Phi(c;p,K;N)$ by applying Lemma 4.6 to the factor $(\mathbf{B}h)(2\pi\sqrt{\omega_1\omega_2}/c)$ on the right-hand side of (5.17); let it be supposed that Lemma 4.6 is applied for

$$\Delta = \left(\left(|c|^2 + N\right) PK\right)^{\zeta} \ , \tag{5.19}$$

where ζ is an arbitrary real number satisfying $0 < \zeta \le 1/5$ (one then has, by the hypotheses of the theorem, $\Delta \ge (NPK)^{\zeta} \ge 1$). The conditions of summation in (5.17) mean that one need only consider $(\mathbf{B}h)(u)$ in cases where $|u|$ lies in the interval $[r,R]$ with endpoints $R = 2\pi\sqrt{N}/|c| > 0$ and $r = R/\sqrt{2}$. Since $\Delta \ge 1$ and $R/r < 2$, there exists an $M \in \mathbb{N}$ such that (4.21) holds for all $u \in \mathbb{C}$ such that $|u| \in [r,R]$. This M (considered fixed henceforth) will certainly satisfy

$$1 < M \asymp \left(1 + |c|^{-1}N^{1/2}\right)\Delta \ . \tag{5.20}$$

After first rewriting the summand on the right-hand side of (5.17) by means of the result (4.22)-(4.25) of Lemma 4.6 (applied for $u = |u|e^{i\theta} = 2\pi\sqrt{\omega_1\omega_2}/c$), and then making a permissible change to the order of summation and integration, one obtains:

$$\begin{aligned}\Phi(c;P,K;N) = {} & \frac{1}{4\pi^3}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{P,K}(\eta,\xi)e^{-\xi^2-\eta^2}\widetilde{\Phi}\left(c,M;N;\psi\left(\frac{\xi}{K}\,,\,\frac{\eta}{P}\,;\phi\right),\phi\right)\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}\phi \; + \\ & + \Phi_M^*(c;P,K;N)\ ,\end{aligned} \tag{5.21}$$

where, with $\theta_z = \mathrm{Arg}(z)$ (for $z \in \mathbb{C}$) and $F_{P,K}(\eta,\xi)$, $\psi(y,x;\phi)$ and $A_M(\phi,\theta)$ as in (4.23) and (4.24), one has

$$\begin{aligned}&\widetilde{\Phi}(c,M;N;\psi,\phi) = \\ &\quad = \sum_{\substack{\omega_1,\omega_2\in\mathfrak{O} \\ N/2<|\omega_1|^2,|\omega_2|^2\le N}}\sum A_M\left(\phi,\frac{\theta_{\omega_1}+\theta_{\omega_2}}{2}-\theta_c\right)\cos\left(\frac{2\pi\psi\sqrt{|\omega_1\omega_2|}}{|c|}\right)S_{\mathfrak{a},\mathfrak{a}}\left(\omega_1,\omega_2;c\right)\overline{b\left(\omega_1\right)}\,b\left(\omega_2\right)\end{aligned} \tag{5.22}$$

and, for $j = 1,2,\ldots$,

$$\begin{aligned}\Phi_M^*(c;P,K;N) = {} & \sum_{\substack{\omega_1,\omega_2\in\mathfrak{O} \\ N/2<|\omega_1|^2,|\omega_2|^2\le N}}\sum \overline{b\left(\omega_1\right)}\,b\left(\omega_2\right)S_{\mathfrak{a},\mathfrak{a}}\left(\omega_1,\omega_2;c\right)E_M\left(P,K;\frac{2\pi\sqrt{\omega_1\omega_2}}{c}\right) \ll_j \\ \ll_j {} & \sum_{\substack{\omega_1,\omega_2\in\mathfrak{O} \\ N/2<|\omega_1|^2,|\omega_2|^2\le N}}\sum \left|b\left(\omega_1\right)b\left(\omega_2\right)S_{\mathfrak{a},\mathfrak{a}}\left(\omega_1,\omega_2;c\right)\right|\left(P^2+K^2\right)\left(1+\frac{N^{1/2}}{|c|}\right)\Delta^{1-2j}\ .\end{aligned} \tag{5.23}$$

For later reference, note here that by (5.18), (5.19) and the Cauchy-Schwarz inequality, the case $j = [1/\zeta]+2$ of (5.23) implies the bounds:

$$\begin{aligned}\Phi_M^*(c;P,K;N) &= O_\zeta\left(|c|^2\left(P^2+K^2\right)\left(1+|c|^{-1}N^{1/2}\right)\left(\left(|c|^2+N\right)PK\right)^{-2}\left(\sum_{\substack{\omega\in\mathfrak{O} \\ N/2<|\omega|^2\le N}}|b(\omega)|\right)^2\right) = \\ &= O_\zeta\left(\left(P^{-2}+K^{-2}\right)|c|\left(|c|+N^{1/2}\right)\left(|c|^2+N\right)^{-2}O(N)\left\|\mathbf{b}_N\right\|_2^2\right) \ll_\zeta \\ &\ll_\zeta \left(P^{-2}+K^{-2}\right)|c|N\left(|c|+N^{1/2}\right)^{-3}\left\|\mathbf{b}_N\right\|_2^2\ .\end{aligned} \tag{5.24}$$

Since $A_M(\phi,\theta)$ is real-valued for $\phi,\theta\in\mathbb{R}$, and since, by (1.5.10) and (1.5.13),

$$\overline{S_{\mathfrak{a},\mathfrak{a}}(\omega_1,\omega_2;c)} = S_{\mathfrak{a},\mathfrak{a}}(-\omega_1,-\omega_2;c) = S_{\mathfrak{a},\mathfrak{a}}(\omega_2,\omega_1;c) \qquad (\omega_1,\omega_2\in\mathfrak{O}),$$

it follows by Euler's identity, $e^{it}=\cos(t)+i\sin(t)$, that one may reformulate (5.22) as:

$$\widetilde{\Phi}(c,M;N;\psi,\phi) = \mathrm{Re}\left(\Phi^{\circ}(c,M;N;\psi,\phi)\right),$$

where

$$\Phi^{\circ}(c,M;N;\psi,\phi) = \\ = \sum\sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\le N}} A_M\!\left(\phi,\frac{\theta_{\omega_1}+\theta_{\omega_2}}{2}-\theta_c\right)\mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c|}\right) S_{\mathfrak{a},\mathfrak{a}}(\omega_1,\omega_2;c)\,\overline{b(\omega_1)}\,b(\omega_2)\,, \tag{5.25}$$

Hence, and by (5.21),

$$|\Phi(c;P,K;N)| \le \frac{1}{4\pi^3}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi^2-\eta^2}\left|F_{P,K}(\eta,\xi)\,\Phi^{\circ}\!\left(c,M;N;\psi\!\left(\frac{\xi}{K},\frac{\eta}{P};\phi\right),\phi\right)\right| \mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}\phi \,+ \\ + |\Phi_M^*(c;P,K;N)| \tag{5.26}$$

(one can in fact show that by omitting both pairs of absolute value parentheses from the upper bound given in (5.26) one obtains, instead of that upper bound, an expression that is identically equal to $\Phi(c;P,K;N)$).

As an alternative to the use of (4.22) in (5.17), one may choose instead to apply in (5.17) the other result (4.26) of Lemma 4.6; then, via steps similar to those that produced (5.28) (and with the same choice of Δ and M as before), one obtains:

$$|\Phi(c;P,K;N)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi^2-\eta^2}G_{P,K}(\eta,\xi)\left|\Phi^{\bullet}\!\left(c,M;N;\psi\!\left(\frac{\xi}{K},\frac{\eta}{P};\phi\right),\phi\right)\right| \mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}\phi \,+ \\ + |\Phi_M^*(c;P,K;N)| \tag{5.27}$$

where $\psi(y,x;\phi)$ and $G_{P,K}(\eta,\xi)$ are as in (4.23) and (4.27), while

$$\Phi^{\bullet}(c,M;N;\psi,\phi) = \\ = \sum\sum_{\substack{\omega_1,\omega_2\in\mathfrak{O}\\ N/2<|\omega_1|^2,|\omega_2|^2\le N}} A_M\!\left(\phi,\frac{\theta_{\omega_1}+\theta_{\omega_2}}{2}-\theta_c\right)\mathrm{e}\!\left(\frac{\psi\sqrt{|\omega_1\omega_2|}}{|c|}\right) S_{\mathfrak{a},\mathfrak{a}}(\omega_1,\omega_2;c)\,\overline{b^{\times}(\omega_1)}\,b^{\times}(\omega_2)\,, \tag{5.28}$$

with $A_M(\phi,\theta)$ as in (4.24) and

$$b^{\times}(\omega) = \left|\frac{\omega}{c}\right| b(\omega) \qquad (0\ne\omega\in\mathfrak{O}), \tag{5.29}$$

and $\Phi_M^*(c;P,K;N)$ is the same term seen in both (5.21) and (5.24). Now, since $\exp(i\theta_z)=z/|z|$ for $0\ne z\in\mathbb{C}$, it is a trivial consequence of (5.25) and (4.24) that

$$|\Phi^{\circ}(c,M;N;\psi,\phi)| \le U_{\mathfrak{a}}(\psi,c;M;N,b) \qquad (\psi,\phi\in\mathbb{R}), \tag{5.30}$$

where $U_{\mathfrak{a}}(\psi,c;M;N,b)$ is as given by equation (1.9.25) of Proposition 2. Similarly, by (5.28) and (4.24),

$$|\Phi^{\bullet}(c,M;N;\psi,\phi)| \le U_{\mathfrak{a}}\left(\psi,c;M;N,b^{\times}\right) \qquad (\psi,\phi\in\mathbb{R}), \tag{5.31}$$

where $U_{\mathfrak{a}}(\psi,c;M;N,b^{\times})$ is given by (1.9.25) with $b^{\times}$ substituted for b throughout.

Given (5.20), and given that $\Delta \geq 1$, it follows from (5.30) and the result (1.9.27) of Proposition 2 that

$$\begin{aligned} \Phi^{\circ}(c, M; N; \psi, \phi) &\ll (1+|\psi|)^{1/2} \left(|c|M + N^{1/2}\right)\left(|c| + N^{1/2}\right) \|\mathbf{b}_N\|_2^2 \ll \\ &\ll (1+|\psi|)^{1/2} \Delta \left(N^{1/2} + |c|\right)^2 \|\mathbf{b}_N\|_2^2 \qquad \text{for } \psi, \phi \in \mathbb{R}\,. \end{aligned} \tag{5.32}$$

Moreover, since (5.29) and (1.9.16) imply

$$\left\|\mathbf{b}_N^{\times}\right\|_2^2 \leq |c|^{-2} N \|\mathbf{b}_N\|_2^2 \ , \tag{5.33}$$

it likewise follows by (5.31) and (1.9.27) (with $b^{\times}$ substituted for b) that

$$\begin{aligned} \Phi^{\bullet}(c, M; N; \psi, \phi) &\ll (1+|\psi|)^{1/2} \Delta \left(|c| + N^{1/2}\right)^2 |c|^{-2} N \|\mathbf{b}_N\|_2^2 \ll \\ &\ll (1+|\psi|)^{1/2} \Delta \left(N^{1/2} + |c|^{-1} N\right)^2 \|\mathbf{b}_N\|_2^2 \qquad \text{for } \psi, \phi \in \mathbb{R}\,. \end{aligned} \tag{5.34}$$

By (4.23) and (4.27),

$$|\psi(y, x; \phi)| \leq 2\cosh(y) \leq 2\exp(|y|) \qquad (\phi, x, y \in \mathbb{R}) \tag{5.35}$$

and

$$0 \leq G_{P,K}(\eta, \xi) \leq 2\left((\xi/K)^2 \cosh^2(\xi/K) + (\eta/P)^2\right) \ll e^{2|\xi|/K} (\xi/K)^2 + (\eta/P)^2 \ , \tag{5.36}$$

so that (given that $K \geq 1$) one has, for $-\pi \leq \phi \leq \pi$,

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi^2-\eta^2} G_{P,K}(\eta, \xi) \left(1 + \left|\psi\left(\frac{\xi}{K}, \frac{\eta}{P}; \phi\right)\right|\right)^{1/2} \mathrm{d}\eta\, \mathrm{d}\xi \ll \\ \ll \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi^2-\eta^2}\left(e^{2|\xi|/K}(\xi/K)^2 + (\eta/P)^2\right) e^{|\xi|/(2K)} \mathrm{d}\eta\, \mathrm{d}\xi \leq \\ \leq 2\left(K^{-2} + P^{-2}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\xi^2 + \eta^2\right) e^{(5/2)\xi - \xi^2 - \eta^2} \mathrm{d}\eta\, \mathrm{d}\xi \ll \\ \ll K^{-2} + P^{-2}\,. \end{aligned} \tag{5.37}$$

Therefore, by applying the estimates (5.24) and (5.34) for terms on the right-hand side of (5.27), one finds (given (5.19), and since $N \geq 1$) that

$$\begin{aligned} \Phi(c; P, K; N) &\ll \left(P^{-2} + K^{-2}\right)\left(\Delta\left(N + |c|^{-2}N^2\right) + O_\zeta\left(|c|^{-2}N\right)\right) \|\mathbf{b}_N\|_2^2 \ll \\ &\ll \left(P^{-2} + K^{-2}\right)(PK)^{\zeta}\left(|c|^2 + O_\zeta(N)\right)^{1+\zeta} |c|^{-2} N \|\mathbf{b}_N\|_2^2 \ll_\zeta \\ &\ll_\zeta (PK)^{\zeta}\left(P^{-2} + K^{-2}\right)\left(N|c|^{2\zeta} + N^{2+\zeta}|c|^{-2}\right) \|\mathbf{b}_N\|_2^2 \ . \end{aligned} \tag{5.38}$$

For a further alternative upper bound on $\Phi(c; P, K; N)$ (useful when $|c|$ is large), one may apply (5.27), and then (5.31) combined with the result (1.9.26) of Proposition 2 (with $b^{\times}$ substituted for b there): given (5.33) and (5.37), and given the upper bound $\tau(c) \ll_\zeta |c|^{2\zeta}$ (which is an elementary corollary of the fundamental theorem of arithmetic for the Gaussian integers), one may in this way obtain the estimate

$$\begin{aligned} \Phi(c; P, K; N) &\ll \left(P^{-2} + K^{-2}\right) \tau^{3/2}(c) |c| M N \left\|\mathbf{b}_N^{\times}\right\|_2^2 + |I_M^*(c; P, K; N)| = \\ &= O_\zeta\Big(\left(P^{-2} + K^{-2}\right)|c|^{2\zeta - 1} M N^2 \|\mathbf{b}_N\|_2^2\Big) + |I_M^*(c; P, K; N)|\,. \end{aligned}$$

Moreover, by (5.19), (5.20) and (5.24), and since $|c|, N, \Delta \geq 1$, it follows from this last bound that one has

$$\begin{aligned} \Phi(c; P, K; N) &= O_\zeta\left(\left(P^{-2} + K^{-2}\right)\left(|c|^{2\zeta-1}\left(1 + |c|^{-1}N^{1/2}\right)\Delta N^2 + |c|^{-2} N\right) \|\mathbf{b}_N\|_2^2\right) \ll_\zeta \\ &\ll_\zeta (PK)^{\zeta}\left(P^{-2} + K^{-2}\right) |c|^{4\zeta - 1} N^2 \left(1 + |c|^{-1} N^{1/2}\right)^{1+2\zeta} \|\mathbf{b}_N\|_2^2 \ . \end{aligned} \tag{5.39}$$

Given that $0 < \zeta \leq 1/5$, one may deduce from the bounds (5.38) and (5.39) that

$$\Phi(c; P, K; N) \ll_\zeta (PK)^\zeta \left(P^{-2} + K^{-2}\right) |c|^{-\zeta} N^{1+5\zeta} \|\mathbf{b}_N\|_2^2 \quad \text{ if } |c|^2 > N^{1-2\zeta} \tag{5.40}$$

(i.e. the bound (5.39) implies this for $|c| > N$, while the bound (5.38) implies this for $N \geq |c| > N^{(1/2)-\zeta}$). Now the sum $\sum_{0 \neq \alpha \in \mathfrak{O}} |\alpha|^{-2\sigma}$ is convergent when $\sigma > 1$, and so, given the definition (1.9.15) and Remark 1.9.7, it follows by the result (1.9.24) of Proposition 2 that

$$\sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}} |c|^{-2\sigma} \leq \sum_{0 \neq \alpha \in \mathfrak{O}} \left| \frac{\alpha}{\mu(\mathfrak{a})} \right|^{-2\sigma} \ll_\sigma |\mu(\mathfrak{a})|^{2\sigma} \leq |\mu(\mathfrak{a})|^2 \qquad (\sigma > 1). \tag{5.41}$$

Therefore application of the bound (5.40) shows that

$$\sum_{\substack{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}} \\ |c|^2 > N^{1-2\zeta}}} \frac{|\Phi(c; P, K; N)|}{|c|^2} \ll_\zeta (PK)^\zeta \left(P^{-2} + K^{-2}\right) N^{1+5\zeta} \|\mathbf{b}_N\|_2^2 \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}} \frac{1}{|c|^{2+\zeta}} \ll_\zeta$$

$$\ll_\zeta (PK)^\zeta \left(P^{-2} + K^{-2}\right) |\mu(\mathfrak{a})|^2 N^{1+5\zeta} \|\mathbf{b}_N\|_2^2 \ . \tag{5.42}$$

Suppose now that

$$0 < |c|^2 \leq N^{1-2\zeta} \ . \tag{5.43}$$

Since $\zeta > 0$ and $N \geq 1$, it is certainly implied by (5.43) that c and N satisfy the case $A_1 = 1$, $\varepsilon = \zeta$ of the condition (1.9.28) of Proposition 2. Indeed, given (5.19), (5.20), (5.30), (5.31), (5.33) and (5.43), the result (1.9.28)-(1.9.30) of Proposition 2 implies that if $\phi, \psi \in \mathbb{R}$ and

$$0 < |\psi| \leq 2e \quad \text{(say)} \tag{5.44}$$

then one has:

$$\begin{aligned} \Phi^\circ(c, M; N; \psi, \phi) &= O_\zeta\left(|\psi|^{-1/2}\right)\left(|c|^{1/2} N^{3/4} + M|c|^{3/2} N^{1/4}\right) N^\zeta \|\mathbf{b}_N\|_2^2 = \\ &= O_\zeta\left(|\psi|^{-1/2}\right)\left(|c|^{1/2} N^{3/4} + \Delta\left(|c|^{3/2} N^{1/4} + |c|^{1/2} N^{3/4}\right)\right) N^\zeta \|\mathbf{b}_N\|_2^2 \ll \\ &\ll O_\zeta\left(|\psi|^{-1/2}\right) \Delta |c|^{1/2} N^{(3/4)+\zeta} \|\mathbf{b}_N\|_2^2 \ll_\zeta \\ &\ll_\zeta |\psi|^{-1/2} (PK)^\zeta N^{(3/4)+2\zeta} |c|^{1/2} \|\mathbf{b}_N\|_2^2 \end{aligned} \tag{5.45}$$

and, similarly,

$$\begin{aligned} \Phi^\bullet(c, M; N; \psi, \phi) &= O_\zeta\left(|\psi|^{-1/2} (PK)^\zeta N^{(3/4)+2\zeta} |c|^{1/2} \left\|\mathbf{b}_N^\times\right\|_2^2\right) \ll_\zeta \\ &\ll_\zeta |\psi|^{-1/2} (PK)^\zeta N^{(7/4)+2\zeta} |c|^{-3/2} \|\mathbf{b}_N\|_2^2 \ . \end{aligned} \tag{5.46}$$

In order to facilitate the application of the last bound one may note that, by (5.27) and (4.27), it is certainly the case that one has

$$|\Phi(c; P, K; N)| \leq \frac{1}{2\pi} \left(\sum_{j=-2}^{2} I_j \right) + \Phi_M^*(c; P, K; N) \ , \tag{5.47}$$

where

$$I_0 = \int_{-K}^{K} \int_{-P}^{P} e^{-\xi^2 - \eta^2} G_{P,K}(\eta, \xi)\, \iota\!\left(\frac{\xi}{K}, \frac{\eta}{P}\right) \mathrm{d}\eta \mathrm{d}\xi \ ,$$

$$I_{\pm 1} = \int_K^\infty \int_{-\infty}^\infty e^{-\xi^2-\eta^2} G_{P,K}(\eta,\xi)\, \iota\biggl(\pm\frac{\xi}{K}, \frac{\eta}{P}\biggr)\, \mathrm{d}\eta \mathrm{d}\xi\ ,$$

$$I_{\pm 2} = \int_{-\infty}^\infty \int_P^\infty e^{-\xi^2-\eta^2} G_{P,K}(\eta,\xi)\, \iota\biggl(\frac{\xi}{K}, \pm\frac{\eta}{P}\biggr)\, \mathrm{d}\eta \mathrm{d}\xi$$

and

$$\iota(y,x) = \int_{-\pi}^{\pi} \bigl|\Phi^{\bullet}\bigl(c, M; N; \psi(y,x;\phi),\phi\bigr)\bigr|\, \mathrm{d}\phi\ .$$

By (4.23) and (5.35), the condition (5.44) will hold for $\psi = \psi(y,x;\phi)$ if $y \in [-1,1]$, $x,\phi \in (-\pi,\pi)$ and $\tan(\phi) \neq \tanh(y)\tan(x)$. Moreover, since $0 < 1 < \pi/2$, it follows by Lemma 4.8 that

$$\int_{-\pi}^{\pi} |\psi(y,x;\phi)|^{-1/2} \mathrm{d}\phi \ll \frac{1}{\sqrt{\cos(1)}} \ll 1 \qquad \text{for } x \in [-1,1] \text{ and } y \in \mathbb{R}\,.$$

Therefore it follows from the bound (5.46) for $\bigl|\Phi^{\bullet}\bigl(c, M; N; \psi(y,x;\phi),\phi\bigr)\bigr|$ that one has:

$$\iota(y,x) \ll_\zeta (PK)^\zeta N^{(7/4)+2\zeta} |c|^{-3/2}\, \|\mathbf{b}_N\|_2^2 \qquad \text{for } x,y \in [-1,1]\,.$$

This, together with the bound obtained in (5.37), enables one to conclude that

$$I_0 \ll_\zeta (PK)^\zeta \left(P^{-2} + K^{-2}\right) N^{(7/4)+2\zeta} |c|^{-3/2}\, \|\mathbf{b}_N\|_2^2\ . \tag{5.48}$$

In estimating $I_{\pm 1}$ and $I_{\pm 2}$ one may use the bound

$$\iota(y,x) \ll \exp(|y|/2)(PKN)^\zeta N^2 |c|^{-2}\, \|\mathbf{b}_N\|_2^2 \qquad (x,y \in \mathbb{R}), \tag{5.49}$$

which follows, given (5.43), from (5.34), (5.35) and (5.19). Indeed, since one has

$$\int_K^\infty \int_{-\infty}^\infty e^{-\xi^2-\eta^2} \left(e^{2\xi/K} \left(\frac{\xi}{K}\right)^2 + \left(\frac{\eta}{P}\right)^2 \right) e^{\xi/(2K)} \mathrm{d}\eta\, \mathrm{d}\xi \ll \left(\frac{1}{P^2} + \frac{1}{K^2}\right) \int_K^\infty e^{(5/2)\xi - \xi^2} \xi^2 \mathrm{d}\xi \leq$$
$$\leq \left(\frac{1}{P^2} + \frac{1}{K^2}\right) \exp\left(-\frac{K^2}{2}\right) \int_{-\infty}^\infty e^{\left(5\xi-\xi^2\right)/2} \xi^2 \mathrm{d}\xi\ ,$$

it follows from (5.49) and (5.36) that

$$I_{\pm 1} \ll \exp\left(-K^2/2\right) (PK)^\zeta \left(P^{-2} + K^{-2}\right) N^{2+\zeta} |c|^{-2}\, \|\mathbf{b}_N\|_2^2\ .$$

Similar reasoning shows that the estimates (5.49) and (5.36) imply also that

$$I_{\pm 2} \ll \exp\left(-P^2/2\right) (PK)^\zeta \left(P^{-2} + K^{-2}\right) N^{2+\zeta} |c|^{-2}\, \|\mathbf{b}_N\|_2^2\ .$$

By (5.24), (5.47), (5.48) and the bounds just obtained for the integrals $I_{\pm 1}$ and $I_{\pm 2}$, it follows that, subject to the condition (5.43) holding, one has:

$$\begin{aligned}\Phi(c;P,K;N) &\ll O_\zeta\Bigl(\left(P^{-2}+K^{-2}\right)\left((PK)^\zeta N^{(7/4)+2\zeta}|c|^{-3/2} + N^{-1/2}|c|\right)\|\mathbf{b}_N\|_2^2\Bigr)\ + \\ &\quad + \bigl(e^{-P^2/2}+e^{-K^2/2}\bigr)(PK)^\zeta\left(P^{-2}+K^{-2}\right)N^{2+\zeta}|c|^{-2}\,\|\mathbf{b}_N\|_2^2 \ll \\ &\ll (PK)^\zeta\bigl(P^{-2}+K^{-2}\bigr)\Bigl(O_\zeta\bigl(N^{(7/4)+2\zeta}|c|^{-3/2}\bigr) + \bigl(e^{-P^2/2}+e^{-K^2/2}\bigr)N^{2+\zeta}|c|^{-2}\Bigr)\|\mathbf{b}_N\|_2^2.\end{aligned} \tag{5.50}$$

Note that (5.50) was derived principally from (5.27) and the bounds for $|\Phi^{\bullet}(c,M;N;\psi,\phi)|$ given by (5.46) and (5.34). By using instead (5.26) and the bounds (5.45) and (5.32) found for $|\Phi^{\circ}(c,M;N;\psi,\phi)|$, one similarly obtains (as an alternative to (5.50)) the estimate

$$\Phi(c;P,K;N) \ll (PK)^\zeta\bigl(P^2+K^2\bigr)\Bigl(O_\zeta\bigl(N^{(3/4)+2\zeta}|c|^{1/2}\bigr) + \bigl(e^{-P^2/2}+e^{-K^2/2}\bigr)N^{1+\zeta}\Bigr)\,\|\mathbf{b}_N\|_2^2\ , \tag{5.51}$$

subject to the condition (5.43) holding.

Since $0 < \zeta \leq 1/5 < 3/2$, it follows by (5.41) that one has:

$$\sum_{\substack{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}} \\ |c| > X}} |c|^{-7/2} < X^{\zeta-(3/2)} \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}} |c|^{-2-\zeta} \ll_\zeta X^{\zeta-(3/2)} |\mu(\mathfrak{a})|^2 \qquad \text{for } X > 0\,.$$

Hence, and with the aid of the case $\sigma = 2$ of (5.41), one may deduce from (5.50) that

$$\begin{aligned}
&\sum_{\substack{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}} \\ N^{(1/2)-\zeta} \geq |c| > \frac{N^{(1/2)-\zeta}}{PK}}} \frac{|\Phi(c;P,K;N)|}{|c|^2} \ll \\
&\qquad \ll (PK)^\zeta \left(\frac{1}{P^2} + \frac{1}{K^2}\right) \Biggl(O_\zeta\Biggl(N^{(7/4)+2\zeta} \left(\frac{N^{(1/2)-\zeta}}{PK}\right)^{\zeta-(3/2)} |\mu(\mathfrak{a})|^2 \Biggr) + \\
&\qquad\qquad + \left(e^{-P^2/2} + e^{-K^2/2}\right) N^{2+\zeta} |\mu(\mathfrak{a})|^4 \Biggr) \|\mathbf{b}_N\|_2^2 \ll_\zeta \\
&\qquad \ll_\zeta (PK)^\zeta \left(\frac{1}{P^2} + \frac{1}{K^2}\right) \Biggl((PK)^{3/2} N^{1+4\zeta} |\mu(\mathfrak{a})|^2 + \\
&\qquad\qquad + \left(e^{-P^2/2} + e^{-K^2/2}\right) N^{2+\zeta} |\mu(\mathfrak{a})|^4 \Biggr) \|\mathbf{b}_N\|_2^2 \,. \qquad (5.52)
\end{aligned}$$

Moreover, since the bound (5.51) shows that if $|c| \leq N^{(1/2)-\zeta}/(PK)$ then

$$|c|^\zeta \Phi(c;P,K;N) \ll (PK)^\zeta \left(P^2 + K^2\right) \left(O_\zeta\left((PK)^{-1/2}\right) + e^{-P^2/2} + e^{-K^2/2} \right) N^{1+2\zeta} \, \|\mathbf{b}_N\|_2^2 \ ,$$

one may therefore use the case $\sigma = 1 + (\zeta/2)$ of (5.41) to deduce from (5.51) that

$$\begin{aligned}
&\sum_{\substack{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}} \\ |c| \leq N^{(1/2)-\zeta}/(PK)}} \frac{|\Phi(c;P,K;N)|}{|c|^2} \ll_\zeta \\
&\qquad \ll_\zeta (PK)^\zeta \left(P^2 + K^2\right) \left((PK)^{-1/2} + e^{-P^2/2} + e^{-K^2/2} \right) N^{1+2\zeta} |\mu(\mathfrak{a})|^2 \, \|\mathbf{b}_N\|_2^2 = \\
&\qquad = (PK)^\zeta \left(\frac{1}{P^2} + \frac{1}{K^2}\right) \Bigl((PK)^{3/2} N^{1+2\zeta} |\mu(\mathfrak{a})|^2 + \\
&\qquad\qquad + \left(e^{-P^2/2} + e^{-K^2/2}\right) (PK)^2 N^{1+2\zeta} |\mu(\mathfrak{a})|^2 \Bigr) \, \|\mathbf{b}_N\|_2^2 \ . \qquad (5.53)
\end{aligned}$$

Observe now that, given the result (1.9.24) of Proposition 2, the sum over c on the left-hand side of (5.53) is in fact an empty sum unless $|\mu(\mathfrak{a})| N^{(1/2)-\zeta}/(PK) \geq 1$. One may therefore replace the upper bound on the right-hand side of (5.53) by

$$O_\zeta \left((PK)^\zeta \left(\frac{1}{P^2} + \frac{1}{K^2}\right) \left((PK)^{3/2} N^{1+2\zeta} |\mu(\mathfrak{a})|^2 + \left(e^{-P^2/2} + e^{-K^2/2}\right) N^2 |\mu(\mathfrak{a})|^4 \right) \|\mathbf{b}_N\|_2^2 \right) .$$

Given this modification of (5.53), together with the complementary bounds found in (5.52) and (5.42), it now follows by the triangle inequality that the sum $\mathcal{L}^{\mathfrak{a}}(q_0, P, K; N, b)$ defined in (5.16) must satisfy

$$\begin{aligned}
&\mathcal{L}^{\mathfrak{a}}(q_0, P, K; N, b) \ll_\zeta \\
&\qquad \ll_\zeta (PKN)^{5\zeta} \left(P^{-2} + K^{-2}\right) \left((PK)^{3/2} N |\mu(\mathfrak{a})|^2 + \left(e^{-P^2/2} + e^{-K^2/2}\right) N^2 |\mu(\mathfrak{a})|^4 \right) \|\mathbf{b}_N\|_2^2 \,. \qquad (5.54)
\end{aligned}$$

Since (5.54) has been established for an arbitrary $\zeta \in (0, 1/5]$, it follows now by (5.4), (5.15) and (5.54) that, for $0 < \eta \leq 1$,

$$\mathcal{E}^{\mathfrak{a}}(q_0, P, K; N, b) = \frac{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]}{8\pi^2} \left(P^2 + K^2\right) PK \left(1 + O\big(P^2 e^{-\pi^2 P^2}\big)\right) \|\mathbf{b}_N^{\mathfrak{a}}\|_2^2 \; + \\ + O_\eta\left((PKN)^\eta \big(P^2 + K^2\big) \left(\frac{N|\mu(\mathfrak{a})|^2}{(PK)^{1/2}} + \big(e^{-P^2/2} + e^{-K^2/2}\big) \frac{N^2 |\mu(\mathfrak{a})|^4}{(PK)^2} \right) \|\mathbf{b}_N\|_2^2 \right). \tag{5.55}$$

Note here that, by (5.10), (1.9.16) and the arithmetic-geometric mean inequality, one has

$$\|\mathbf{b}_N^{\mathfrak{a}}\|_2^2 \leq \|\mathbf{b}_N\|_2^2 \; .$$

Therefore if one supposes now that $j \in \{0, 1\}$, then it may be deduced from (5.2), (5.8) and (5.55) that, for $P_1, K_1 \geq 1$ and $0 < \eta \leq 1$, one has:

$$E_j^{\mathfrak{a}}(q_0, P_1, K_1; N, b) \ll \\ \ll \big(P_1^2 + K_1^2\big) \left(P_1 K_1 + O_\eta\big((P_1 K_1 N)^\eta\big) \left(\frac{N|\mu(\mathfrak{a})|^2}{(P_1 K_1)^{1/2}} + \big(e^{-P_1} + e^{-K_1}\big) \frac{N^2 |\mu(\mathfrak{a})|^4}{(P_1 K_1)^2} \right) \right) \|\mathbf{b}_N\|_2^2 \; . \tag{5.56}$$

By the definition (1.9.15) and Remark 1.9.7, one has $1/\mu(\mathfrak{a}) \in \mathfrak{O}$, so that $0 < |\mu(\mathfrak{a})| \leq 1$. Therefore, and since $P, K \geq 1$, it follows by (5.56) for $\eta = 1/2$, $P_1 = P$ and $K_1 = K$ that one has (in particular):

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \ll \big(P^2 + K^2\big) \left(PK + N^{3/2} + (PK)^{-3/2} N^{5/2} \right) \|\mathbf{b}_N\|_2^2 \; . \tag{5.57}$$

The aim now is to show that (5.57) and (5.56) (for any given $\eta \in (0, 1]$) imply the bound

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \ll \big(P^2 + K^2\big) \left(PK + O_\eta\left(\frac{N^{1 + 4\sqrt{\eta}} \, |\mu(\mathfrak{a})|^2}{(PK)^{1/2}} \right) \right) \|\mathbf{b}_N\|_2^2 \; . \tag{5.58}$$

This (since (5.56) holds for all $\eta \in (0, 1]$) will be enough to prove the theorem: for, in cases where $0 < \varepsilon \leq 4$, the bound (1.9.14) follows immediately from (5.58) for $\eta = (\varepsilon/4)^2$; while, in cases where $\varepsilon > 4$, the bound (1.9.14) is (given that $N \geq 1$) a trivial corollary of (1.9.14) for $\varepsilon = 4$.

Now since (5.57) implies the desired result (5.58) if $PK > N^{3/2}$, one may henceforth suppose that

$$PK \leq N^{3/2} \; . \tag{5.59}$$

Independently of the conclusion just reached, one may observe also that if one has

$$P_1 \geq P, \quad K_1 \geq K \quad \text{and} \quad P_1 K_1 \leq N^2 \; , \tag{5.60}$$

then, by (1.9.12)-(1.9.13) and (5.56), it follows that

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \leq E_j^{\mathfrak{a}}(q_0, P_1, K_1; N, b) \ll \\ \ll \big(P_1^2 + K_1^2\big) \left(P_1 K_1 + O_\eta\big(N^{3\eta}\big) \left(\frac{N|\mu(\mathfrak{a})|^2}{(PK)^{1/2}} + \big(e^{-P_1} + e^{-K_1}\big) \frac{N^2 |\mu(\mathfrak{a})|^4}{(PK)^2} \right) \right) \|\mathbf{b}_N\|_2^2 . \tag{5.61}$$

Given (5.59), the condition (5.60) is (in particular) satisfied when $P_1 = P$ and $K_1 = K$, so that (5.61) holds in this case. Hence, on noting that $e^{-P} + e^{-K} \leq 2/e$, one obtains the bound

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \ll \big(P^2 + K^2\big) \left(PK + O_\eta\big(N^{3\eta}\big) \left(\frac{N|\mu(\mathfrak{a})|^2}{(PK)^{1/2}} + \frac{N^2 |\mu(\mathfrak{a})|^4}{(PK)^2} \right) \right) \|\mathbf{b}_N\|_2^2 , \tag{5.62}$$

which (since $PK \geq 1$, $N \geq 1$ and $\sqrt{\eta} \geq \eta > 0$) implies the desired result (5.58) if $N|\mu(\mathfrak{a})|^2 < N^{\sqrt{\eta}}$. Therefore it is only cases where one has

$$N|\mu(\mathfrak{a})|^2 \geq N^{\sqrt{\eta}}$$

that require any further consideration. In these cases $N^\eta \leq \left(N|\mu(\mathfrak{a})|^2\right)^{\sqrt{\eta}}$, so that, by (5.62), one obtains:

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \ll \left(P^2 + K^2\right)\left(PK + O_\eta\left(\frac{\left(N|\mu(\mathfrak{a})|^2\right)^{1+3\sqrt{\eta}}}{(PK)^{1/2}} + \frac{\left(N|\mu(\mathfrak{a})|^2\right)^{2+3\sqrt{\eta}}}{(PK)^2}\right)\right)\|\mathbf{b}_N\|_2^2 \ .$$

Since $N \geq 1 \geq |\mu(\mathfrak{a})|$, the bound just noted certainly implies the desired result (5.58) if $(PK)^{3/2} > N|\mu(\mathfrak{a})|^2$, and so one may henceforth suppose that

$$PK \leq \frac{N|\mu(\mathfrak{a})|^2}{(PK)^{1/2}} \ . \tag{5.63}$$

Take now

$$P_1 = \max\left\{P, \left(N|\mu(\mathfrak{a})|^2\right)^{\eta/2}\right\} \quad \text{and} \quad K_1 = \max\left\{K, \left(N|\mu(\mathfrak{a})|^2\right)^{\eta/2}\right\} \ .$$

Since $N, P, K \geq 1$ and $\left(N|\mu(\mathfrak{a})|^2\right)^{\eta/2} \leq N^{\eta/2} \leq N^{1/2}$, it follows by (5.59) that (5.60) holds. Therefore (5.61) also holds. Moreover, since

$$e^{-P_1} + e^{-K_1} \leq 2\exp\left(-\left(N|\mu(\mathfrak{a})|^2\right)^{\eta/2}\right) \ll_\eta \left(N|\mu(\mathfrak{a})|^2\right)^{-1} \quad \text{and} \quad PK \geq 1 \ ,$$

it follows from (5.61) that

$$E_j^{\mathfrak{a}}(q_0, P, K; N, b) \ll \left(P_1^2 + K_1^2\right)\left(P_1 K_1 + O_\eta\left(\frac{N^{1+3\eta}|\mu(\mathfrak{a})|^2}{(PK)^{1/2}}\right)\right)\|\mathbf{b}_N\|_2^2 \ .$$

Now observe that this implies the desired result (5.58): for one has, in the above bound,

$$P_1^2 + K_1^2 \leq \left(N|\mu(\mathfrak{a})|^2\right)^\eta \left(P^2 + K^2\right) \leq N^\eta \left(P^2 + K^2\right)$$

and, by (5.63),

$$P_1 K_1 \leq \left(N|\mu(\mathfrak{a})|^2\right)^\eta PK \leq \frac{\left(N|\mu(\mathfrak{a})|^2\right)^{1+\eta}}{(PK)^{1/2}} \leq \frac{N^{1+\eta}|\mu(\mathfrak{a})|^2}{(PK)^{1/2}} \ ,$$

where $N^\eta \leq N^{\sqrt{\eta}}$ (given that $0 < \eta \leq 1$ and $N \geq 1$). Since no other cases remain to be considered, it has therefore now been shown that, subject to the hypotheses of the theorem, the bound (5.58) holds for $j \in \{0, 1\}$ and $0 < \eta \leq 1$. For the reasons mentioned below (5.58), this completes proof of the theorem ■

§6. Appendix on the proof of the sum formula.

In this appendix we give a description of the proof of Theorem B. The proof we shall describe is obtained through an adaptation of the work [5] of Bruggeman and Motohashi; it also owes much to Lokvenec-Guleska's thesis [32], in which a significant generalisation of the Bruggeman-Motohashi sum formula is obtained.

It is to be assumed throughout this appendix that q_0 is a given non-zero Gaussian integer, and that $\Gamma = \Gamma_0(q_0)$ (the Hecke congruence subgroup of $SL(2, \mathbb{Z}[i])$ defined in Equation (1.1.1)). Notations already introduced in Section 1 of the paper remain in use: we shall define additional terminology as the need arises, and our useage of any such additional terminology shall not be limited to the subsection of the appendix in which the relevant definition is stated (in particular, a full understanding of Subsection 6.5 and Subsection 6.6 requires some familiarity with terminology defined in the first four subsections of this appendix).

§6.1 Generalised Kloosterman sums.

In this subsection we aim to justify what is stated in and between (1.5.11) and (1.5.12), concerning the generalised Kloosterman sums, and associated sets ${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$, defined in (1.5.8)-(1.5.10). For our proof of the upper bound (1.5.12) we shall need the analogue, due to Bruggeman and Miatello [4], of the 'Weil-Estermann' bound for classical Kloosterman sums. We need also the following lemma, the proof of which is modelled very closely on Motohashi's work, in Section 15 of [36], on Hecke congruence sugroups of $SL(2,\mathbb{Z})$.

Lemma 6.1.1. *Let $u_1, w_1, u_2, w_2 \in \mathfrak{O}$ satisfy*

$$w_j \mid q_0 \qquad \text{and} \qquad (u_j, w_j) \sim 1 \qquad\qquad (j = 1, 2). \tag{6.1.1}$$

Let $\mathfrak{a}'$ and $\mathfrak{b}'$ be the cusps of $\Gamma = \Gamma_0(q_0) \le SL(2,\mathfrak{O})$ given by $\mathfrak{a}' = u_1/w_1$ and $\mathfrak{b}' = u_2/w_2$. Suppose moreoever that the scaling matrices, $g_{u_1/w_1} = g_{\mathfrak{a}'}$ and $g_{u_2/w_2} = g_{\mathfrak{b}'}$ are chosen similarly to $g_{u/w}$ in the proof of Lemma 4.2, so that one has:

$$g_{u_j/w_j} = \varpi_{u_j/w_j}\tau_{v_j} \in SL(2,\mathbb{C}) \qquad\qquad (j = 1, 2), \tag{6.1.2}$$

where

$$\varpi_{u/w} = \begin{pmatrix} u & -\tilde{w} \\ w & \tilde{u} \end{pmatrix} \in SL(2,\mathfrak{O}) \qquad\qquad (u, w \in \mathfrak{O},\ w \neq 0 \text{ and } (u,w) \sim 1), \tag{6.1.3}$$

with $\tilde{u}, \tilde{w}$ denoting an arbitrary pair of Gaussian integers such that $u\tilde{u} + w\tilde{w} = 1$, while

$$\tau_v = \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix} \in SL(2,\mathbb{C}) \qquad\qquad (v \in \mathbb{C}^*), \tag{6.1.4}$$

and v_1, v_2 are an arbitrary pair of Gaussian integers such that

$$v_j \sim \frac{q_0/w_j}{(q_0/w_j\,,\, w_j)} \qquad\qquad (j = 1, 2). \tag{6.1.5}$$

Then $g_{\mathfrak{a}'}, g_{\mathfrak{b}'} \in SL(2,\mathbb{C})$ are such that the conditions (1.1.16) and (1.1.20)-(1.1.21) are satisfied when either $\mathfrak{c} = \mathfrak{a}'$ or $\mathfrak{c} = \mathfrak{b}'$. The corresponding set ${}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'}$ (for which see (1.5.8)-(1.5.9)) satisfies

$${}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'} \subseteq \sqrt{v_1 v_2}\,\mathfrak{O} - \{0\} \tag{6.1.6}$$

(with v_1, v_2 as in (6.1.5)) and, for all $m, n \in \mathfrak{O}$ and all $C \in \mathfrak{O} - \{0\}$ such that $\sqrt{v_1 v_2}\, C \in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'}$, one has

$$S_{\mathfrak{a}',\mathfrak{b}'}(m, n; C\sqrt{v_1 v_2}) = \sum_{\substack{A \bmod v_1 C\mathfrak{O},\ D \bmod v_2 C\mathfrak{O} \\ AD \equiv 1 \bmod C\mathfrak{O}}} \chi_{q_0}\left(\varpi_{\mathfrak{a}'} g(A, D; C)\varpi_{\mathfrak{b}'}^{-1}\right) \mathrm{e}\left(\mathrm{Re}\left(\frac{mA}{v_1 C} + \frac{nD}{v_2 C}\right)\right), \tag{6.1.7}$$

where the matrices $\varpi_{\mathfrak{a}'} = \varpi_{u_1/w_1}$ and $\varpi_{\mathfrak{b}'} = \varpi_{u_2/w_2}$ are as in (6.1.2)-(6.1.3), while

$$g(a, d; c) = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \qquad (a, d \in \mathbb{C},\ c \in \mathbb{C}^*), \tag{6.1.8}$$

and where, for $g \in SL(2,\mathbb{C})$,

$$\chi_{q_0}(g) = \begin{cases} 1 & \text{if } g \in \Gamma_0(q_0), \\ 0 & \text{otherwise.} \end{cases} \tag{6.1.9}$$

In particular, the terms of the sum on the right-hand side of (6.1.7) are well defined, so that, for $a, d \in \mathbb{C}$, and all relevant choices of C, one has

$$\chi_{q_0}\left(\varpi_{\mathfrak{a}'} g(a + s v_1 C, d + t v_2 C; C)\varpi_{\mathfrak{b}'}^{-1}\right) = \chi_{q_0}\left(\varpi_{\mathfrak{a}'} g(a, d; C)\varpi_{\mathfrak{b}'}^{-1}\right) \quad \text{when } s, t \in \mathfrak{O}. \tag{6.1.10}$$

Proof. The scaling matrices $g_{\mathfrak{a}'}, g_{\mathfrak{b}'}$ are chosen similarly to the scaling matrix $g_{u/w}$ which features in the proof of Lemma 4.2. That proof shows $g_{u/w}$ to be such that the conditions (1.1.16) and (1.1.20)-(1.1.21) are satisfied when $\mathfrak{c} = u/w$. One may therefore infer that $g_{\mathfrak{a}'}$ and $g_{\mathfrak{b}'}$ are such that the same is true both for $\mathfrak{c} = \mathfrak{a}'$, and for $\mathfrak{c} = \mathfrak{b}'$.

By (6.1.2) and (6.1.3), one has $g_{\mathfrak{a}'}^{-1}\Gamma g_{\mathfrak{b}} \subseteq \tau_{v_1}^{-1}\varpi_{u_1/w_1}^{-1} SL(2,\mathfrak{O})\varpi_{u_2/w_2}\tau_{v_2} \subseteq \tau_{v_1}^{-1}SL(2,\mathfrak{O})\tau_{v_2}$. Given this, and the definitions (6.1.4), (1.5.8) and (1.5.9), a very short calculation suffices to show that (6.1.6) holds.

We now have only to prove the results in (6.1.7)-(6.1.10). We may suppose that m and n are Gaussian integers, and that $\sqrt{v_1 v_2}\, C = c \in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'}$. By (6.1.6), we have $0 \neq C \in \mathfrak{O}$. Given that the relevant cases of (1.1.20)-(1.1.21) hold, it follows by the definition (1.5.10) that

$$S_{\mathfrak{a}',\mathfrak{b}'}(m,n;c) = \sum_{\substack{g \in B^+\backslash g_{\mathfrak{a}'}^{-1}\, {}^{\mathfrak{a}'}\Gamma^{\mathfrak{b}'}(c) g_{\mathfrak{b}'}/B^+ \\ g = \begin{pmatrix} \alpha(g) & * \\ c & \delta(g)\end{pmatrix}}} \mathrm{e}\left(\mathrm{Re}\left(m\,\frac{\alpha(g)}{c} + n\,\frac{\delta(g)}{c}\right)\right), \tag{6.1.11}$$

where the summation is such that g runs over the elements of a complete set of representatives, $\left\{g^{(r)} : r \in R\right\}$ (say), for the set of double cosets $B^+\backslash g_{\mathfrak{a}'}^{-1}\, {}^{\mathfrak{a}'}\Gamma^{\mathfrak{b}'}(c) g_{\mathfrak{b}'}/B^+ = B^+\backslash \tau_{v_1}^{-1}\varpi_{\mathfrak{a}'}^{-1}\, {}^{\mathfrak{a}'}\Gamma^{\mathfrak{b}'}(c)\varpi_{\mathfrak{b}'}\tau_{v_2}/B^+$. The corresponding set $\left\{\tau_{v_1} g^{(r)}\tau_{v_2}^{-1} : r \in R\right\}$ is, given (6.1.4), a complete set of representatives for the set of double cosets $B_1^+\backslash \varpi_{\mathfrak{a}'}^{-1}\, {}^{\mathfrak{a}'}\Gamma^{\mathfrak{b}'}(c)\varpi_{\mathfrak{b}'}/B_2^+$, where, for $j = 1,2$,

$$B_j^+ = \tau_{v_j} B^+ \tau_{v_j}^{-1} = \left\{h\left[\sqrt{v_j}\right] n[\xi] h\left[1/\sqrt{v_j}\right] : \xi \in \mathfrak{O}\right\} = \left\{n\left[v_j\xi\right] : \xi \in \mathfrak{O}\right\}. \tag{6.1.12}$$

Hence, upon rewriting the summation in (6.1.11) in terms of

$$\tilde{g} = \tau_{v_1} g \tau_{v_2}^{-1} = \begin{pmatrix} \alpha(g)\sqrt{v_1/v_2} & * \\ c/\sqrt{v_1 v_2} & \delta(g)\sqrt{v_2/v_1}\end{pmatrix} = \begin{pmatrix} A(\tilde{g}) & * \\ C & D(\tilde{g})\end{pmatrix} \quad \text{(say)},$$

one finds that

$$S_{\mathfrak{a}',\mathfrak{b}'}(m,n;C\sqrt{v_1 v_2}) = \sum_{\substack{\tilde{g} \in B_1^+\backslash \varpi_{\mathfrak{a}'}^{-1}\, {}^{\mathfrak{a}'}\Gamma^{\mathfrak{b}'}(C\sqrt{v_1 v_2})\, \varpi_{\mathfrak{b}'}/B_2^+ \\ \tilde{g} = \begin{pmatrix} A(\tilde{g}) & * \\ C & D(\tilde{g})\end{pmatrix}}} \mathrm{e}\left(\mathrm{Re}\left(m\,\frac{A(\tilde{g})}{v_1 C} + n\,\frac{D(\tilde{g})}{v_2 C}\right)\right), \tag{6.1.13}$$

where the set $\varpi_{\mathfrak{a}'}^{-1}\, {}^{\mathfrak{a}'}\Gamma^{\mathfrak{b}'}\big(C\sqrt{v_1 v_2}\big)\, \varpi_{\mathfrak{b}'}$ is (of course) invariant under both multiplication on the left by elements of B_1^+, and multiplication on the right by elements of B_2^+, and is moreover a subset of $SL(2,\mathfrak{O})$ (for, by (1.5.8), and (6.1.1) and (6.1.3), we have ${}^{\mathfrak{a}'}\Gamma^{\mathfrak{b}'}\big(C\sqrt{v_1 v_2}\big) \subset \Gamma \leq SL(2,\mathfrak{O})$ and $\varpi_{\mathfrak{a}'}, \varpi_{\mathfrak{b}'} \in SL(2,\mathfrak{O})$). By the last observation, we have, in the above, $A(\tilde{g}), D(\tilde{g}) \in \mathfrak{O}$, and, given that $C \neq 0$, may express $\tilde{g}$ as $g(A(\tilde{g}), D(\tilde{g}); C)$, where the notation $g(a,d;c)$ is defined in (6.1.8). Furthermore, by (6.1.8) it is evident that, for $z_1, z_2 \in \mathbb{C}$, one has

$$n\left[z_1\right] g(A,D;C) n\left[z_2\right] = n\left[z_1 + \frac{A}{C}\right] g(0,0;C) n\left[\frac{D}{C} + z_2\right] = g\left(A + Cz_1, D + Cz_2; C\right)\ , \tag{6.1.14}$$

and so the result stated in (6.1.7)-(6.1.9) is merely a more explicit formulation of (6.1.13) and (6.1.12).

Finally, given (6.1.12) and (6.1.14), it suffices for proof of the result (6.1.10) that we can show $\varpi_{\mathfrak{a}'} B_1^+ \varpi_{\mathfrak{a}'}^{-1}$ and $\varpi_{\mathfrak{b}'} B_2^+ \varpi_{\mathfrak{b}'}^{-1}$ to be subgroups of the group $\Gamma = \Gamma_0(q_0)$. This is easily achieved. Indeed, by the first equality of (6.1.12), and by (6.1.2) and (1.1.20), one has

$$\varpi_{u_j/w_j} B_j^+ \varpi_{u_j/w_j}^{-1} = \varpi_{u_j/w_j}\tau_{v_j} B^+ \tau_{v_j}^{-1}\varpi_{u_j/w_j}^{-1} = g_{u_j/w_j} B^+ g_{u_j/w_j}^{-1} = \Gamma'_{u_j/w_j} < \Gamma \qquad (j = 1,2),$$

which (given that $u_1/w_1 = \mathfrak{a}'$ and $u_2/w_2 = \mathfrak{b}'$) is just what was required ■

Corollary 6.1.2. *Subject to the hypotheses of the above lemma, one has the trivial bound:*

$$|S_{\mathfrak{a}',\mathfrak{b}'}(m,n;c)| \leq |c|^2\,|v_1v_2| \qquad\qquad (m,n\in\mathfrak{O} \text{ and } c\in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'}). \tag{6.1.15}$$

Proof. In the sum on the right-hand side of equation (6.1.7) there are $|(\mathfrak{O}/C\mathfrak{O})^*| \leq |C|^2$ choices for $A \bmod C\mathfrak{O}$; and when that residue class is given, the congruence condition $AD \equiv 1 \bmod C\mathfrak{O}$ determines $D \bmod C\mathfrak{O}$, so that there remain just $|v_1|^2$ choices for $A \bmod v_1C\mathfrak{O}$ and $|v_2|^2$ choices for $D \bmod v_2C\mathfrak{O}$. Hence the sum appearing in (6.1.7) contains at most $|Cv_1v_2|^2$ non-zero terms. The bound (6.1.15) therefore follows from (6.1.6) and the case $C=c/\sqrt{v_1v_2}$ of (6.1.7)-(6.1.9), by way of the triangle inequality ■

The next lemma is the Gaussian integer analogue of Equation (15.14) of [36]. Before stating the lemma, it is helpful to clarify that it assumes a more special choice of all the relevant scaling matrices than was the case in Lemma 6.1.1. To be precise, although it is still to be assumed that $g_{u/w}=\varpi_{u/w}\tau_v$, with $\varpi_{u/w}$ and τ_v as in (6.1.3)-(6.1.5), it is now also to be supposed that $\varpi_{u/w}$ is chosen in such a way that the relevant Gaussian integer $\tilde{u}$ in (6.1.3) satisfies

$$u\tilde{u} \equiv 1 \bmod q_0\mathfrak{O}\ . \tag{6.1.16}$$

Since this is only possible when u is coprime to q_0, the choice of cusps is also more restricted than is the case in Lemma 6.1.1.

Lemma 6.1.3. *Let the hypotheses of Lemma 6.1.1 be satisfied. Suppose moreover that $(u_1u_2,q_0)\sim 1$, and that the scaling matrices $g_{u_1/w_1}=g_{\mathfrak{a}'}$ and $g_{u_2/w_2}=g_{\mathfrak{b}'}$ satisfy the additional constraint imposed in (6.1.16). Let C be a non-zero Gaussian integer such that $C\sqrt{v_1v_2}\in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'}$. Then, for some $C_{q_0}, C'_{q_0}, \tilde{C}_{q_0}\in\mathfrak{O}$, one has*

$$C_{q_0}C'_{q_0}=C\ , \qquad C'_{q_0}\sim (C,q_0^{\infty})\ , \qquad C_{q_0}\tilde{C}_{q_0}\equiv 1 \bmod \left[[v_1,v_2]\,C'_{q_0},q_0\right]\mathfrak{O} \tag{6.1.17}$$

(where $[r,s]$ denotes an arbitrary least common multiple of r and s, and where the notation 'q_0^{∞}' signifies that one has $C'_{q_0}\sim(C,q_0^N)$ for all sufficiently large $N\in\mathbb{N}$) and also:

$${}^{\mathfrak{a}''}\mathcal{C}^{\mathfrak{b}''} \ni C'_{q_0}\sqrt{v_1v_2} \tag{6.1.18}$$

and, for all $m,n\in\mathfrak{O}$,

$$S_{\mathfrak{a}',\mathfrak{b}'}(m,n;C\sqrt{v_1v_2}) = S_{\mathfrak{a}'',\mathfrak{b}''}\Big(\tilde{C}_{q_0}m,\tilde{C}_{q_0}n;C'_{q_0}\sqrt{v_1v_2}\Big)\, S\Big(\left(C'_{q_0}v_1\right)^*m,\left(C'_{q_0}v_2\right)^*n;C_{q_0}\Big)\ , \tag{6.1.19}$$

where

$$\mathfrak{a}''=\tilde{C}_{q_0}u_1/w_1\ , \qquad\qquad \mathfrak{b}''=\tilde{C}_{q_0}u_2/w_2 \tag{6.1.20}$$

and the associated scaling matrices are $g_{\mathfrak{a}''}=\varpi_{\tilde{C}_{q_0}u_1/w_1}\tau_{v_1}$ and $g_{\mathfrak{b}''}=\varpi_{\tilde{C}_{q_0}u_2/w_2}\tau_{v_2}$ (with τ_v, v_1 and v_2 as in (6.1.4)-(6.1.5), and with $\varpi_{u/w}$ as indicated in (6.1.3) and (6.1.16)), while $S(a,b;c)$ denotes the 'simple Kloosterman sum' defined in (2.16), and the $*$-notation has the significance explained in our subsection on notation (under the sub-heading 'Other Number-Theoretic Notation').

Proof. Since the proof involves calculations very similar to those of Section 15 of [36], we only sketch the main points.

Note firstly that, since $\mathfrak{O}$ is a unique factorisation domain, we can certainly find a Gaussian integer C'_{q_0} satisfying the condition $C'_{q_0}\sim(C,q_0^{\infty})$ in (6.1.17). Then $C'_{q_0}\mid C$, and the Gaussian integer $C_{q_0}=C/C'_{q_0}$ is coprime to each of C'_{q_0}, q_0, v_1 and v_2 (the latter two being factors of q_0, by virtue of (6.1.5) and (6.1.1)). Hence there exist $C_{q_0}, C'_{q_0}, \tilde{C}_{q_0}\in\mathfrak{O}$ such that all the conditions in (6.1.17) are satisfied.

Let $C_{q_0}, C'_{q_0}, \tilde{C}_{q_0}\in\mathfrak{O}$ satisfy the conditions in (6.1.17), and let the cusps $\mathfrak{a}'',\mathfrak{b}''$ be given by (6.1.20). Since $(u_1u_2,q_0)\sim 1$ (by hypothesis), $(\tilde{C}_{q_0},q_0)\sim 1$ (by (6.1.17)) and $w_1,w_2\mid q_0$ (by (6.1.1)), we may choose the scaling matrices $g_{\mathfrak{a}''}$ and $g_{\mathfrak{b}''}$ to be as described below (6.1.20). Then, given the hypotheses of the lemma

concerning $g_{\mathfrak{a}'}$ and $g_{\mathfrak{b}'}$, it may be shown by a calculation that, with regard to the result (6.1.7) of Lemma 6.1.1, one has:

$$\chi_{q_0}\left(\varpi_{\mathfrak{a}'} g(A,D;C)\varpi_{\mathfrak{b}'}^{-1}\right) = \chi_{q_0}\left(\varpi_{\mathfrak{a}''} g\left(A,D;C'_{q_0}\right)\varpi_{\mathfrak{b}''}^{-1}\right) \qquad \text{for } \ A,D\in\mathfrak{O} \ \text{ with } \ AD\equiv 1 \bmod C\mathfrak{O} \tag{6.1.21}$$

(where $\varpi_{u/w}$, $g(a,d;c)$ and χ_{q_0} are given by (6.1.3), (6.1.8) and (6.1.9)). We remark that the full force of the congruence in (6.1.17) is not required in the above: it would suffice there to have just $C_{q_0}\tilde{C}_{q_0} \equiv 1 \bmod q_0\mathfrak{O}$.

Given that ${}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'} \ni C\sqrt{v_1v_2}$, and given the assumptions made concerning scaling matrices, it may be deduced from (6.1.21) that ${}^{\mathfrak{a}''}\mathcal{C}^{\mathfrak{b}''} \ni C'_{q_0}\sqrt{v_1v_2}$, so that one may apply Lemma 6.1.1 with $\mathfrak{a}''$, $\mathfrak{b}''$ and C'_{q_0} substituted for $\mathfrak{a}'$, $\mathfrak{b}'$ and C, respectively (and with no change in the values of v_1 and v_2). By making these substitutions in (6.1.10), one finds that the right-hand side of the equation in (6.1.21) is a function of the residue class of $A \bmod v_1C'_{q_0}\mathfrak{O}$ and the residue class of $D \bmod v_2C'_{q_0}\mathfrak{O}$. Therefore, and since $(v_1C'_{q_0}, C_{q_0}) \sim 1$ and $(v_2C'_{q_0}, C_{q_0}) \sim 1$, one may apply the Chinese Remainder Theorem to deduce from (6.1.7) and (6.1.21) that

$$S_{\mathfrak{a}',\mathfrak{b}'}(m,n;C\sqrt{v_1v_2}) = XY \ ,$$

where

$$X = \sum_{\substack{A \bmod v_1C'_{q_0}\mathfrak{O},\ D \bmod v_2C'_{q_0}\mathfrak{O} \\ AD\equiv 1 \bmod C'_{q_0}\mathfrak{O}}} \chi_{q_0}\left(\varpi_{\mathfrak{a}''} g\left(A,D;C'_{q_0}\right)\varpi_{\mathfrak{b}''}^{-1}\right) \mathrm{e}\left(\mathrm{Re}\left(\frac{\tilde{C}_{q_0}mA}{v_1C'_{q_0}} + \frac{\tilde{C}_{q_0}nD}{v_2C'_{q_0}}\right)\right)$$

and

$$Y = \sum_{\substack{A \bmod C_{q_0}\mathfrak{O},\ D \bmod C_{q_0}\mathfrak{O} \\ AD\equiv 1 \bmod C_{q_0}\mathfrak{O}}} \mathrm{e}\left(\mathrm{Re}\left(\frac{\left(v_1C'_{q_0}\right)^* mA}{C_{q_0}} + \frac{\left(v_2C'_{q_0}\right)^* nD}{C_{q_0}}\right)\right) = S\Big(\left(C'_{q_0}v_1\right)^* m, \left(C'_{q_0}v_2\right)^* n; C_{q_0}\Big) \ .$$

The result (6.1.19) follows: for, by Lemma 6.1.1 (applied with $\mathfrak{a}''$ and $\mathfrak{b}''$ substituted for $\mathfrak{a}'$ and $\mathfrak{b}'$, respectively), one has $X = S_{\mathfrak{a}'',\mathfrak{b}''}\left(\tilde{C}_{q_0}m, \tilde{C}_{q_0}n; C'_{q_0}\sqrt{v_1v_2}\right)$ ■

Corollary 6.1.4. *Let the combined hypotheses of Lemma 6.1.1 and Lemma 6.1.3 be satisfied. Then, for all $m,n\in\mathfrak{O}$, one has*

$$\left|S_{\mathfrak{a}',\mathfrak{b}'}(m,n;C\sqrt{v_1v_2})\right| \le 2^{3/2}\tau(C)\left|(m,n,C)C\left(C,q_0^\infty\right)v_1^2v_2^2\right| \ , \tag{6.1.22}$$

where $\tau(k)$ equals the number of Gaussian integer divisors of k, and 'q_0^∞' has the same meaning as in (6.1.17).

Proof. By (6.1.20), one has $\mathfrak{a}'' = u'_1/w_1$ and $\mathfrak{b}'' = u'_2/w_2$, where $u'_j = \tilde{C}_{q_0}u_j \in \mathfrak{O}$ $(j=1,2)$, so that, by (6.1.1) and (6.1.17), $(u'_j, w_j) \sim 1$ for $j=1,2$. Hence (and given the result in (6.1.18)) Corollary 6.1.2 may be applied with $\mathfrak{a}''$ and $\mathfrak{b}''$ substituted for $\mathfrak{a}'$ and $\mathfrak{b}'$ (respectively), and with v_1 and v_2 unchanged. Consequently one has the upper bound

$$\left|S_{\mathfrak{a}'',\mathfrak{b}''}\left(\tilde{C}_{q_0}m, \tilde{C}_{q_0}n; C'_{q_0}\sqrt{v_1v_2}\right)\right| \le \left|C'_{q_0}\sqrt{v_1v_2}\right|^2 |v_1v_2| = \left|C'_{q_0}v_1v_2\right|^2 \ . \tag{6.1.23}$$

By the result (2.18) of Lemma 2.4 (the 'Weil-Estermann' bound obtained by Bruggeman and Miatello in [4]), we have also the upper bound

$$\left|S\Big(\left(C'_{q_0}v_1\right)^* m, \left(C'_{q_0}v_2\right)^* n; C_{q_0}\Big)\right| \le 2^{3/2}\tau(C_{q_0})\left|(m,n,C_{q_0})\,C_{q_0}\right| \tag{6.1.24}$$

for the second factor on the right-hand side of equation (6.1.19) (note that we have used here the fact that, in equation (6.1.19), both $(C'_{q_0}v_1)^*$ and $(C'_{q_0}v_2)^*$ are, by definition, coprime to C_{q_0}). By (6.1.24), (6.1.23), (6.1.19) and (6.1.17), the bound (6.1.22) follows ■

The next (and final) lemma in this subsection contains the results stated in (1.5.11) and (1.5.12). In those results, and in the lemma, we make use of the notation '$m_{\mathfrak{c}}$' introduced below equation (1.1.22). Hence, for each cusp $\mathfrak{c}$ of the group $\Gamma = \Gamma_0(q_0) \leq SL(2,\mathfrak{O})$, we take $m_{\mathfrak{c}}$ to signify an arbitrary Gaussian integer satisfying

$$m_{\mathfrak{c}} \sim \begin{cases} q_0/(w^2,q_0) & \text{if } \mathfrak{c} = u/w \text{ with } u,w \in \mathfrak{O},\ w \neq 0 \text{ and } (u,w) \sim 1; \\ 1 & \text{if } \mathfrak{c} = \infty. \end{cases} \tag{6.1.25}$$

(see (1.5.8)-(1.5.10) for the definitions of the relevant generalised Kloosterman sums $S_{\mathfrak{a},\mathfrak{b}}(m,n;c)$ and the associated sets ${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$). It follows from (6.1.25) that the ideal $m_{\mathfrak{c}}\mathfrak{O}$ depends only on the Γ-equivalence class of the cusp $\mathfrak{c}$ (we skip the easy proof of this). For the notations '$\tau(k)$' and 'q_0^{∞}', see below (6.1.17) or (6.1.22).

Lemma 6.1.5. *Let $m,n \in \mathfrak{O}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma = \Gamma_0(q_0)$, and let the associated scaling matrices $g_{\mathfrak{a}}, g_{\mathfrak{b}} \in SL(2,\mathbb{C})$ be such that the conditions (1.1.16) and (1.1.20)-(1.1.21) are satisfied for $\mathfrak{c} \in \{\mathfrak{a},\mathfrak{b}\}$. Then one has, for some $\epsilon \in \mathfrak{O}^*$, both the relation*

$${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}} \subseteq \sqrt{\epsilon m_{\mathfrak{a}} m_{\mathfrak{b}}}\ \mathfrak{O} - \{0\} \tag{6.1.26}$$

and, for all $c = \sqrt{\epsilon m_{\mathfrak{a}} m_{\mathfrak{b}}}\ C \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$, the upper bounds

$$|S_{\mathfrak{a},\mathfrak{b}}(m,n;c)| \leq 2^{3/2}\tau(C)\left|(m,n,C)\, C\, (C,q_0^{\infty})\, m_{\mathfrak{a}}^2 m_{\mathfrak{b}}^2\right| \tag{6.1.27}$$

and

$$|S_{\mathfrak{a},\mathfrak{b}}(m,n;c)| \leq |C m_{\mathfrak{a}} m_{\mathfrak{b}}|^2\ . \tag{6.1.28}$$

Proof. By the result (2.4) of Lemma 2.2, there exist $u_1, w_1, u_2, w_2 \in \mathfrak{O}$ satisfying (6.1.1), and such that

$$\mathfrak{a} \overset{\Gamma}{\sim} u_1/w_1 \qquad \text{and} \qquad \mathfrak{b} \overset{\Gamma}{\sim} u_2/w_2\ . \tag{6.1.29}$$

It may at the same time be assumed (in the above) that the hypothesis $(u_1u_2,q_0) \sim 1$ of Lemma 6.1.3 is satisfied: note that this does not entail any loss of generality, for one has $u_j/w_j \sim (u_j + kw_j)/w_j$, for $j = 1,2$ and all $k \in \mathfrak{O}$, and the coprimality conditions in (6.1.1) imply that both of the sets $\{u_1 + kw_2 : k \in \mathfrak{O}\}$ and $\{u_2 + kw_2 : k \in \mathfrak{O}\}$ do contain elements coprime to q_0.

We now write $u_1/w_1 = \mathfrak{a}'$ and $u_2/w_2 = \mathfrak{b}'$, and take the associated scaling matrices, $g_{\mathfrak{a}'} = g_{u_1/w_1}$ and $g_{\mathfrak{b}'} = g_{u_2/w_2}$, to be of the form indicated in (6.1.2)-(6.1.5) of Lemma 6.1.1, so that $g_{\mathfrak{a}'} = \varpi_{u_1/w_1}\tau_{v_1}$ and $g_{\mathfrak{b}'} = \varpi_{u_2/w_2}\tau_{v_2}$ where v_1, v_2, τ_{v_1} and τ_{v_2} are as in (6.1.5) and (6.1.4), while ϖ_{u_1/w_1} and ϖ_{u_2/w_2} are as indicated by (6.1.3). Since $(u_1u_2,q_0) \sim 1$, and since $w_1, w_2 \mid q_0$, we may suppose moreover that, for $j = 1,2$, one has

$$\varpi_{u_j/w_j} = \begin{pmatrix} u_j & -\tilde{w}_j \\ w_j & \tilde{u}_j \end{pmatrix} \in SL(2,\mathfrak{O})\ ,$$

with $\tilde{u}_j \in \mathfrak{O}$ such that $u_j\tilde{u}_j \equiv 1 \bmod q_0\mathfrak{O}$ (i.e. one can find $r_j, s_j \in \mathfrak{O}$ such that $u_jr_j + q_0s_j = 1$, and so may take $\tilde{u}_j = r_j$ and $\tilde{w}_j = (q_0/w_j)s_j$ in the above). Note that (as is stated in Lemma 6.1.1) such a choice of the scaling matrices $g_{\mathfrak{a}'}$ and $g_{\mathfrak{b}'}$ ensures that the conditions (1.1.16) and (1.1.20)-(1.1.21) are satisfied when $\mathfrak{c} \in \{\mathfrak{a}',\mathfrak{b}'\}$. Therefore, given the relations in (6.1.29), it follows by Lemma 2.1, (2.2) and (2.3), that, for some pair of units $\epsilon_1, \epsilon_2 \in \mathfrak{O}^*$, one has both

$${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}} = \sqrt{\epsilon_1\epsilon_2}\ {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'} \tag{6.1.30}$$

and, for all $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$,

$$|S_{\mathfrak{a},\mathfrak{b}}(m,n;c)| = |S_{\mathfrak{a}',\mathfrak{b}'}(\overline{\epsilon_1}\, m, \overline{\epsilon_2}\, n; c/\sqrt{\epsilon_1\epsilon_2})| \tag{6.1.31}$$

(note that, by (1.5.13), the choice of square roots in the above is immaterial).

Now, by (6.1.30) and the result (6.1.6) of Lemma 6.1.1, we have

$${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}} \subseteq \sqrt{\epsilon_1\epsilon_2 v_1 v_2}\ \mathfrak{O} - \{0\}\ , \tag{6.1.32}$$

where the Gaussian integers v_1, v_2 satisfy (6.1.5), and so (see (6.1.25)) are such that $v_1 \sim m_{\mathfrak{a}'}$ and $v_2 \sim m_{\mathfrak{b}'}$. Given the relations in (6.1.29), we have moreover $m_{\mathfrak{a}} \sim m_{\mathfrak{a}'} \sim v_1$ and $m_{\mathfrak{b}} \sim m_{\mathfrak{b}'} \sim v_2$, and so, by (6.1.32), we obtain the result (6.1.26) with $\epsilon = (v_1/m_{\mathfrak{a}})(v_2/m_{\mathfrak{b}})\epsilon_1\epsilon_2 \in \mathfrak{O}^*$. Moreover, with ϵ as just indicated, it follows from (6.1.30) and (6.1.32) that, if $c, C \in \mathbb{C}^*$ are such that $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ and $\sqrt{\epsilon m_{\mathfrak{a}} m_{\mathfrak{b}}}\, C = c$, then one has $0 \neq C \in \mathfrak{O}$ and $\sqrt{v_1 v_2}\, C = c/\sqrt{\epsilon_1\epsilon_2} \in {}^{\mathfrak{a}'}\mathcal{C}^{\mathfrak{b}'}$, and so obtains, by (6.1.31) and Corollary 6.1.4,

$$|S_{\mathfrak{a},\mathfrak{b}}(m,n;c)| = |S_{\mathfrak{a}',\mathfrak{b}'}(\overline{\epsilon_1}\, m, \overline{\epsilon_2}\, n; \sqrt{v_1 v_2}\ C)| \leq 2^{3/2}\tau(C) \left|(\overline{\epsilon_1}\, m, \overline{\epsilon_2}\, n, C)\ C\,(C, q_0^{\infty})\, v_1^2 v_2^2\right| \ .$$

Since we have here $\epsilon_1, \epsilon_2 \in \mathfrak{O}^*$, $v_1 \sim m_{\mathfrak{a}}$ and $v_2 \sim m_{\mathfrak{b}}$, the result (6.1.27) follows immediately. The proof of (6.1.28) is similar, differing only in that one uses Corollary 6.1.2 in place of the 'Weil-Estermann' bound, (6.1.22) ■

§6.2 Poincaré series.

Convergence, continuity and a lemma on inner products. For $\omega \in \mathfrak{O}$, we define $C^0(N\backslash G, \omega)$ to be the space of continuous functions $h : G \to \mathbb{C}$ with the property that, for $n \in N$ and $g \in G$, one has $h(ng) = \psi_\omega(n)h(g)$ (where ψ_ω is the character of N defined in (1.4.3)). We remark that for each $\omega \in \mathfrak{O}$ one has the relation $C^0(N\backslash G, \omega) \subset C^0(B^+\backslash G)$, where

$$C^0\big(B^+\backslash G\big) = \big\{\phi \in C^0(G) \,:\, \phi(bg) = \phi(g) \ \text{ for } \ b \in B^+,\, g \in G\big\}\,.$$

Assume now (and for the remainder of this subsection) that $\mathfrak{a}$ is some cusp of Γ, that $\omega \in \mathfrak{O}$, and that one has $f = f_\omega \in C^0(N\backslash G, \omega)$. Then the conditions (1.1.16) and (1.1.20)-(1.1.21), for $\mathfrak{c} = \mathfrak{a}$, suffice to ensure that if, for all $g \in G$, the series on the right-hand side of the equation (1.5.4) converges absolutely then (1.5.4) defines a function $P^{\mathfrak{a}} f : G \to \mathbb{C}$ which is Γ-automorphic.

In determining sufficient conditions for the absolute convergence of the Poincaré series $P^{\mathfrak{a}} f$ we shall generalise the approach taken in Section 7.1 of [32], where only the case $\mathfrak{a} = \infty$ is considered. As a matter of notational convenience, we begin by defining

$$\rho(na[r]k) = r \qquad \text{for } n \in N,\ r > 0 \text{ and } k \in K, \tag{6.2.1}$$

so that, for each $\nu \in \mathbb{C}$, one has

$$(\rho(g))^{1+\nu} = \varphi_{0,0}(\nu, 0)(g) \qquad (g \in G), \tag{6.2.2}$$

where $\varphi_{\ell,q}(\nu, p) : G \to \mathbb{C}$ is the function defined in (1.3.2). The following lemma will prove useful.

Lemma 6.2.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of Γ, and let $g_{\mathfrak{a}}, g_{\mathfrak{b}} \in G = SL(2,\mathbb{C})$ be such that the conditions (1.1.16) and (1.1.20)-(1.1.21) are satisfied for $\mathfrak{c} \in \{\mathfrak{a}, \mathfrak{b}\}$. Then the family of sets $({}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c))_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}} \cup \{0\}}$, defined by (1.5.8) and (1.5.9), is a partitioning of the set of elements of Γ. For $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}} \cup \{0\}$, $\gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)$ and $g \in G$, one has*

$$\rho\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g\right) = \rho(g) \qquad \textit{if } c = 0; \tag{6.2.3}$$

$$\rho\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g\right)\rho(g) \leq |c|^{-2} \leq |m_{\mathfrak{a}} m_{\mathfrak{b}}|^{-1} \qquad \textit{if } 0 \neq c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}, \tag{6.2.4}$$

where $|m_{\mathfrak{c}}|^2 \in \mathbb{N}$ is the 'width' of the cusp $\mathfrak{c}$ (as defined below (1.1.22)).

Proof. Given that $\{g_{\mathfrak{a}}, g_{\mathfrak{b}}\} \cup \Gamma \subset G = SL(2,\mathbb{C})$, the assertion concerning the partitioning of Γ is an immediate corollary of the definitions in (1.5.8) and (1.5.9).

By (1.5.8), one has $\gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0)$ if and only if $\gamma \in \Gamma$ and $g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}\infty = \infty$. Since the latter relation is equivalent to the relation $\gamma\mathfrak{b} = \mathfrak{a}$, it therefore follows by Equation (2.1) of Lemma 2.1 that if $\gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0)$ then $g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} = h[\eta]n[\beta]$, for some $\eta, \beta \in \mathbb{C}$ with $\eta^2 \in \mathfrak{O}^*$. The result (6.2.3) follows: for, by (1.1.4), (1.1.3), (1.1.9) and (6.2.1), one has $\rho(h[u]n[\beta]g) = |u|^2\rho(n[\beta]g) = |u|^2\rho(g)$ for $u \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, $g \in G$.

Supposing now that $g \in G$, and that $\gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)$ for some non-zero $c \in \mathbb{C}$, it follows by (1.1.4) and the definition (1.5.8) that $\rho(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g) = \rho(g)/(|cz+d|^2 + |c|^2\rho^2(g))$, where $z \in \mathbb{C}$ is the Iwasawa 'z-coordinate'

of g, while $-d/c = g_{\mathfrak{b}}^{-1}\gamma^{-1}g_{\mathfrak{a}}\infty \in \mathbb{C}$. This shows, since $|cz+d|^2 \geq 0$ (while $\rho(g)$ and $|c|$ are strictly positive), that one has $\rho(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}g) \leq 1/(|c|^2\rho(g)) \in (0,\infty)$, which implies the first inequality in (6.2.4). What remains of (6.2.4) is immediate from (6.1.25) and the result (6.1.26) of Lemma 6.1.5 ■

We now suppose that, for some $\sigma_0 > 1$ and some $R_0 > 0$ (both of which may depend on f_ω), one has

$$f_\omega(g) \ll_{f_\omega} (\rho(g))^{1+\sigma_0} \qquad \text{for all } g \in G \text{ such that } \rho(g) \leq R_0. \tag{6.2.5}$$

Note that, since $|f_\omega| \in C^0(N\backslash G, 0)$, it is implied by the hypothesis (6.2.5) that if $R > 0$ then

$$f_\omega(g) \ll_{f_\omega,\sigma_0,R_0,R} (\rho(g))^{1+\sigma_0} \qquad \text{for all } g \in G \text{ such that } \rho(g) \leq R. \tag{6.2.6}$$

Lemma 6.2.2. *Let $\mathfrak{a}$ be a cusp of Γ, let $g_{\mathfrak{a}} \in G$ be such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} = \mathfrak{a}$, let $\omega \in \mathfrak{O}$, $\sigma_0 > 1$ and $R_0 > 0$, and let $f_\omega \in C^0(N\backslash G, \omega)$ be such that the condition (6.2.5) is satisfied. Then, when $J \subseteq G$ is compact, the series $\sum_{\Gamma'_{\mathfrak{a}}\backslash\Gamma} |f_\omega(g_{\mathfrak{a}}^{-1}\gamma g)|$ is uniformly convergent for all $g \in J$.*

Proof. Let J be a compact subset of the set of elements of G. By (6.1.25) and the case $\mathfrak{b} = \infty$, $g_{\mathfrak{b}} = h[1]$ of Lemma 6.2.1, it follows that, for $g \in G$ and $\gamma \in \Gamma$, one has $\rho(g_{\mathfrak{a}}^{-1}\gamma g) \leq \exp(|\log\rho(g)|)$. Since it is moreover the case that the compactness of J implies the existence of $\max\{\exp(|\log\rho(g)|) : g \in J\}$, and since the hypothesis (6.2.5) implies that we have (6.2.6) when (in particular) R is equal to this maximum, we therefore find that

$$f_\omega\left(g_{\mathfrak{a}}^{-1}\gamma g\right) \ll_{f_\omega,\sigma_0,R_0,J} (\rho(g))^{1+\sigma_0} = \varphi_{0,0}\left(\sigma_0,0\right)(g) \qquad \text{for } g \in J,\ \gamma \in \Gamma \tag{6.2.7}$$

(the final equality having been noted in (6.2.2)). It follows that the series $\sum_{\Gamma'_{\mathfrak{a}}\backslash\Gamma} |f_\omega(g_{\mathfrak{a}}^{-1}\gamma g)|$ converges uniformly, for $g \in J$, if and only if the same is true of the series $E^{\mathfrak{a}}_{0,0}(\sigma_0,0)(g)$ defined in (1.8.1). The latter series is an instance of the Eisenstein series extensively discussed in Chapter 3 of [11], and would there be designated by the notation '$E_{g_{\mathfrak{a}}^{-1}}(gj,\sigma_0)$' (in which '$j$' signifies the point $(0,1) \in \mathbb{H}_3$). Given (6.2.2), it is shown by the case '$\eta = \infty$, $B = I$' of Proposition 3.2.3 of [11], and by [11], Definition 3.1.2, Proposition 3.1.3 and Corollary 3.1.6 (which imply that the 'abscissa of convergence' of the cofinite subgroup $\Gamma \leq SL(2,\mathbb{C})$ equals 1), that if $\alpha > 0$ and $\beta > 1$ then the series

$$\sum_{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma} \left(\frac{\rho\left(g_{\mathfrak{a}}^{-1}\gamma g\right)}{\rho(g)}\right)^{1+\nu} = \sum_{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma} (\rho(g))^{-1-\nu}\,\varphi_{0,0}\left(\nu,0\right)\left(g_{\mathfrak{a}}^{-1}\gamma g\right)$$

converges uniformly for all $(g,\nu) \in G\times\mathbb{C}$ satisfying both $\rho(g) \geq \alpha$ and $\mathrm{Re}(\nu) \geq \beta$; in particular, since $\sigma_0 > 1$, and since the compactness of J implies that one has $\min\{\rho(g) : g \in J\} > 0$, this series converges uniformly for all $(g,\nu) \in J \times \{\sigma_0\}$. It follows that the series $E^{\mathfrak{a}}_{0,0}(\sigma_0,0)(g) = \sum_{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma}[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]^{-1}\varphi_{0,0}(\sigma_0,0)(g_{\mathfrak{a}}^{-1}\gamma g)$ is (likewise) uniformly convergent for $g \in J$: for one has $\min\{(\rho(g))^{-1-\sigma_0} : g \in J\} > 0$ (by virtue of the continuity of the function $\rho : G \to (0,\infty)$ and the compactness of J), and the factor $[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]^{-1} \in (0,1/2]$ is independent of g. This, with (6.2.7), completes the proof of the lemma ■

Corollary 6.2.3. *If the hypotheses of the above lemma are satisfied, and if $f = f_\omega$, then the equation (1.5.4) defines a Γ-automorphic function $P^{\mathfrak{a}}f : G \to \mathbb{C}$ which is continuous on G.*

Proof. By the hypothesis that $f_\omega \in C^0(N\backslash G, \omega)$, each term $f_\omega(g_{\mathfrak{a}}^{-1}\gamma g)$ in the the series in (1.5.4) is a continuous function of g. Therefore, given that G is locally compact, it follows by the uniform convergence established in the lemma that the series in (1.5.4) is convergent for all $g \in G$, and has a sum that is a continuous function of g. The absolute convergence of the series in (1.5.4) implies that all rearrangements of that series have the same sum, and so the Γ-automorphicity of Pf_ω may be seen to follow by observing that, when $\tau_0 \in \Gamma$ is given, the mapping $\Gamma'_{\mathfrak{a}}\gamma \mapsto \Gamma'_{\mathfrak{a}}\gamma\tau_0$ is a permutation on the set of right cosets of $\Gamma'_{\mathfrak{a}}$ in Γ (this implying that if $\mathcal{X}$ is a complete set of right coset representatives of $\Gamma'_{\mathfrak{a}}$ in Γ then so too is $\mathcal{X}\tau$, if $\tau \in \Gamma$) ■

The next lemma is of a well-known type: it is, in particular, a minor extension of Lemma 7.3.1 of [32] (which dealt only with Poincaré series associated with the cusp ∞). We shall later obtain an extension of it (Lemma 6.6.2) through which the key part that certain Poincaré series have to play in the proof of the sum formula (Theorem B) is mediated. Before stating the lemma we clarify that henceforth $C^0(\Gamma\backslash G)$ denotes the space of those functions $F : G \rightarrow \mathbb{C}$ that are both continuous and Γ-automorphic, while $L^1(\Gamma\backslash G)$ will denote the space of those measurable and Γ-automorphic functions $f : G \rightarrow \mathbb{C}$ that satisfy $\int_{\Gamma\backslash G} |f| \, \mathrm{d}g < \infty$.

Lemma 6.2.4. *Let the hypotheses of Lemma 6.2.2 be satisfied. Suppose, moreover, that $\phi \in C^0(\Gamma\backslash G)$ is such that $(P^{\mathfrak{a}}|f_\omega|) \cdot \phi \in L^1(\Gamma\backslash G)$. Then*

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \, \langle P^{\mathfrak{a}} f_\omega, \phi \rangle_{\Gamma\backslash G} = \langle f_\omega, F^{\mathfrak{a}}_\omega \phi \rangle_{N\backslash G} \ , \tag{6.2.8}$$

where, for $f, F \in C^0(N\backslash G, \omega)$,

$$\langle f, F \rangle_{N\backslash G} = \int_{N\backslash G} f(g) \, \overline{F(g)} \, \mathrm{d}\dot{g} = \int_A \int_K \frac{f(ak) \, \overline{F(ak)} \, \mathrm{d}k \, \mathrm{d}a}{(\rho(a))^2} = \int_0^\infty \int_K \frac{f\left(a[r]k\right) \ \overline{F\left(a[r]k\right)} \, \mathrm{d}k \, \mathrm{d}r}{r^3} \ . \tag{6.2.9}$$

Proof. Let $\{\gamma_j : j \in \mathbb{N}\}$ be a complete set of representatives of the right cosets of $\Gamma'_{\mathfrak{a}}$ in Γ. Then, by the triangle inequality and (1.5.4), it follows that

$$\left| \sum_{j=1}^{J} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma_j g\right) \overline{\phi(g)} \right| \leq [\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \left(P^{\mathfrak{a}} \left|f_\omega\right|\right)(g) \left|\phi(g)\right| \qquad \text{for all } J \in \mathbb{N} \text{ and all } g \in G.$$

This, combined with Lemma 6.2.2 and the hypothesis that $(P^{\mathfrak{a}}|f_\omega|) \cdot \phi \in L^1(\Gamma\backslash G)$, enables the application of Lebesgue's 'dominated convergence' theorem, so that one obtains:

$$\int_{\Gamma\backslash G} [\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \left(P^{\mathfrak{a}} f_\omega\right)(g) \, \overline{\phi(g)} \, \mathrm{d}g = \lim_{J\to\infty} \int_{\Gamma\backslash G} \sum_{j=1}^{J} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma_j g\right) \overline{\phi(g)} \, \mathrm{d}g = \sum_{\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma} \int_{\Gamma\backslash G} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma g\right) \overline{\phi(g)} \, \mathrm{d}g \ .$$

Hence, given that ϕ is Γ-automorphic, one finds (by the usual 'unfolding' method) that

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \int_{\Gamma\backslash G} \left(P^{\mathfrak{a}} f_\omega\right)(g) \, \overline{\phi(g)} \, \mathrm{d}g = \sum_{\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma} \int_{\gamma\mathcal{F}_{\Gamma\backslash G}} f_\omega\left(g_{\mathfrak{a}}^{-1} g\right) \overline{\phi(g)} \, \mathrm{d}g \ , \tag{6.2.10}$$

where $\mathcal{F}_{\Gamma\backslash G}$ denotes a (measurable) fundamental domain for $\Gamma\backslash G$, while $\gamma\mathcal{F}_{\Gamma\backslash G} = \{\gamma g : g \in \mathcal{F}_{\Gamma\backslash G}\}$.

One may repeat the above steps with $|f_\omega|$ and $|\phi|$ substituted for f_ω and ϕ (respectively). Consequently, and by virtue of the countable additivity of the relevant integrals, it follows that the measurable function $g \mapsto \left|f_\omega(g_{\mathfrak{a}}^{-1} g) \, \overline{\phi(g)}\,\right| \in [0, \infty)$ is integrable over $\Gamma'_{\mathfrak{a}}\backslash G$ (with respect to the measure induced by $\mathrm{d}g$); the same is therefore true of the measurable function $g \mapsto f_\omega(g_{\mathfrak{a}}^{-1} g) \, \overline{\phi(g)}$. This shows that the integrals on the right-hand side of Equation (6.2.10) are countably additive, so that (6.2.10) implies the equality

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \int_{\Gamma\backslash G} \left(P^{\mathfrak{a}} f_\omega\right)(g) \, \overline{\phi(g)} \, \mathrm{d}g = \int_{\Gamma'_{\mathfrak{a}}\backslash G} f_\omega\left(g_{\mathfrak{a}}^{-1} g\right) \overline{\phi(g)} \, \mathrm{d}g \ . \tag{6.2.11}$$

If $\mathcal{F}_{\mathfrak{a}}$ is any fundamental domain for $\Gamma'_{\mathfrak{a}}\backslash G$, then (by virtue of the hypothesis that (1.1.20) holds for $\mathfrak{c} = \mathfrak{a}$) the set $g_{\mathfrak{a}}^{-1}\mathcal{F}_{\mathfrak{a}} = \{g_{\mathfrak{a}}^{-1} g : g \in \mathcal{F}_{\mathfrak{a}}\} \subset G$ is a fundamental domain for $B^+\backslash G$, where $B^+ < N$ is given by (1.1.21). Therefore it follows that, by virtue of the left invariance of the Haar measure $\mathrm{d}g$, by the hypothesis that $f_\omega \in C^0(N\backslash G, \omega)$, and by the Iwasawa decomposition $G = NAK$, one has:

$$\begin{aligned} \int_{\Gamma'_{\mathfrak{a}}\backslash G} f_\omega\left(g_{\mathfrak{a}}^{-1} g\right) \overline{\phi(g)} \, \mathrm{d}g &= \int_{B^+\backslash G} f_\omega\left(g\right) \overline{\phi\left(g_{\mathfrak{a}} g\right)} \, \mathrm{d}g = \\ &= \int_{B^+\backslash N} \int_A \int_K \psi_\omega(n) f_\omega(ak) \, \overline{\phi\left(g_{\mathfrak{a}} n a k\right)} \left(\rho(a)\right)^{-2} \mathrm{d}k \, \mathrm{d}a \, \mathrm{d}n \ , \end{aligned} \tag{6.2.12}$$

where $\rho : G \to (0,\infty)$ is given by (6.2.1), and the Haar measures $\mathrm{d}n$, $\mathrm{d}a$ and $\mathrm{d}k$ are as in (1.1.10). By Fubini's theorem, one may change the order of integration in the final iterated integral in (6.2.12), so as to give priority to the integration with respect to the variable n; given the definitions of $F_\omega^{\mathfrak{c}} f$ and $\psi_\omega(n)$ in (1.4.2) and (1.4.3), one thereby obtains from (6.2.11) and (6.2.12) the results stated in (6.2.8)-(6.2.9) ■

Fourier expansions at cusps. Suppose now that $\mathfrak{a}$, $\mathfrak{b}$, $g_{\mathfrak{a}}$, $g_{\mathfrak{b}}$, ω, σ_0, R_0 and f_ω satisfy the combined hypotheses of Lemma 6.2.1 and Lemma 6.2.2. Then it follows by Corollary 6.2.3 that, for each $\omega' \in \mathfrak{O}$, one may define the Fourier term of order ω' for $P^{\mathfrak{a}} f_\omega$ at $\mathfrak{b}$ to be the function $F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega : G \to \mathbb{C}$ given by

$$\left(F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega\right)(g) = \int_{B^+\backslash N} \left(\psi_{\omega'}(n)\right)^{-1} \left(P^{\mathfrak{a}} f_\omega\right)(g_{\mathfrak{b}} n g)\,\mathrm{d}n \qquad (g \in G). \tag{6.2.13}$$

Note that this accords with the definition given in (1.4.2), though there we dealt only with the Fourier expansions of functions lying in the space $C^\infty(\Gamma\backslash G)$; our present hypotheses do not even imply that $P^{\mathfrak{a}} f_\omega$ is differentiable on G, nor do they imply that the Fourier series $\sum_{\omega'\in\mathfrak{O}} (F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega)(g)$ is convergent for all $g \in G$ (see Section 13.41 of [43] for a relevant example). Therefore our present hypotheses do not suffice to ensure that the Fourier expansion (1.4.1) is valid, for all $g \in G$, when one substitutes for $\mathfrak{c}$ and f (there) the cusp $\mathfrak{b}$ and function $P^{\mathfrak{a}} f_\omega$, respectively.

We shall address the question of the representation of Poincaré series by their Fourier expansions (at cusps) in an ad hoc manner, and only as the need arises: see Remark 6.2.6 and the proof of the result (6.5.76) of Lemma 6.5.14, below.

Regardless of whether or not it is the case that $\sum_{\omega'\in\mathfrak{O}} F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega = ((P^{\mathfrak{a}} f_\omega)|\mathfrak{b})$, it does follow from the definitions (6.2.13) and (1.5.4) that, for $\omega' \in \mathfrak{O}$, one has

$$\begin{aligned}\left(F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega\right)(g) &= \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma_{\mathfrak{a}}']} \int\limits_{B^+\backslash N} \sum_{\gamma\in\Gamma_{\mathfrak{a}}'\backslash\Gamma} \left(\psi_{\omega'}(n)\right)^{-1} f_\omega\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} n g\right)\mathrm{d}n = \\ &= \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma_{\mathfrak{a}}']} \sum_{\gamma\in\Gamma_{\mathfrak{a}}'\backslash\Gamma} \int\limits_{B^+\backslash N} \left(\psi_{\omega'}(n)\right)^{-1} f_\omega\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} n g\right)\mathrm{d}n\,. \end{aligned}\tag{6.2.14}$$

Note that $B^+\backslash N$ has a compact fundamental domain, namely the set $\{n[z] : -1/2 \le \mathrm{Re}(z), \mathrm{Im}(z) \le 1/2\}$, so that the uniform convergence established in Lemma 6.2.2 justifies the term by term integration by which the final equality in (6.2.14) is obtained.

Lemma 6.2.5. *Let the combined hypotheses of Lemma 6.2.1 and Lemma 6.2.2 be satisfied, and let $\omega' \in \mathfrak{O}$. Then (1.5.4) and (6.2.13) define a function $F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega$ which lies in the space $C^0(N\backslash G, \omega')$. For $g \in G$ the formula for $\left(F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega\right)(g)$ implied by the case $\mathfrak{a}' = \mathfrak{b}$ of (1.5.5)-(1.5.10) holds, and the sums and integrals occurring (explicitly or implicitly) in the equation (1.5.5) are absolutely convergent.*

Proof. By Corollary 6.2.3 and the case $\mathfrak{c} = \mathfrak{b}$ of (1.1.20), the function $g \mapsto (P^{\mathfrak{a}} f_\omega)(g_{\mathfrak{b}} g)$ is continuous on G, and satisfies $(P^{\mathfrak{a}} f_\omega)(g_{\mathfrak{b}} b g) = (P^{\mathfrak{a}} f_\omega)(g_{\mathfrak{b}} g)$, for $g \in G$ and $b \in B^+$. It follows that, for all $r_1, r_2 \in (0,\infty)$, the function $g \mapsto (P^{\mathfrak{a}} f_\omega)(g_{\mathfrak{b}} g)$ is uniformly continuous on the set $\{na[r]k : n \in N, r_1 \le r \le r_2, k \in K\}$. This fact, combined with the continuity of $(\psi_{\omega'}(n))^{-1}$ and the fact that $B^+\backslash N$ is compact (and of finite measure with respect to $\mathrm{d}n$), is sufficient to establish both the existence and continuity (as a function of g) of the integral on the right-hand side of Equation (6.2.13): we may conclude that (1.5.4) and (6.2.13) define a continuous function $F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega : G \to \mathbb{C}$. Moreover, since the measure $\mathrm{d}n$ on N induces a Haar measure on the group $B^+\backslash N$ it is an immediate consequence of (6.2.13) and (1.4.3) that $(F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega)(ng) = \psi_{\omega'}(n)(F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega)(g)$ for all $g \in G$ and all $n \in N$; this completes the proof that one has $F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega \in C^0(N\backslash G, \omega')$.

Let $g \in G$. Our approach to the proof of the formula (1.5.5) is modelled on Section 7.2 of [32] (in which the case $\mathfrak{a} = \mathfrak{b} = \infty$ is treated); this differs from the approach taken on Pages 39 and 40 of [5], where the case $\Gamma = SL(2,\mathfrak{O})$ of (1.5.5) is obtained via the Poisson summation formula. We observe firstly that, by

(6.2.14) and the partitioning of Γ noted in Lemma 6.2.1, and by (1.1.20) (for $\mathfrak{c} = \mathfrak{b}$) and (6.2.5), one has

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \left(F^{\mathfrak{b}}_{\omega'} P^{\mathfrak{a}} f_\omega \right)(g) = \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0)} \int\limits_{B^+ \backslash N} (\psi_{\omega'}(n))^{-1} f_\omega \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} n g \right) \mathrm{d}n + \\ + \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c) / \Gamma'_{\mathfrak{b}}} \int\limits_N (\psi_{\omega'}(n))^{-1} f_\omega \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} n g \right) \mathrm{d}n \tag{6.2.15}$$

(we have used here the fact that if $0 \neq c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$, and if $\mathcal{Z}$ is a complete set of representatives for the set of double cosets $\{\Gamma'_{\mathfrak{a}} \gamma \Gamma'_{\mathfrak{b}} : \gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)\}$, then the family of sets $(\gamma \Gamma'_{\mathfrak{b}})_{\gamma \in \mathcal{Z}}$ is a partitioning of a complete set of representatives for the set of right cosets $\{\Gamma'_{\mathfrak{a}} \gamma : \gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)\}$).

With regard to the first sum on the right-hand side of (6.2.15), we recall, from the proof of Lemma 6.2.1, that ${}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0) = \{\gamma \in \Gamma : \gamma \mathfrak{b} = \mathfrak{a}\}$, and that for each $\gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0)$ there is some $\eta = \eta(\gamma) \in \mathbb{C}$, with $\eta^2 \in \mathfrak{O}^*$ and some $\beta = \beta(\gamma) \in \mathbb{C}$ such that $h[\eta] n[\beta] = g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}}$, so that $g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} n[z] = n[\eta^2 z] g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}}$ for $z \in \mathbb{C}$. Hence, by the hypothesis that $f_\omega \in C^0(N \backslash G, \omega)$, and by (1.4.3), we find that

$$\sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0)} \int\limits_{B^+ \backslash N} (\psi_{\omega'}(n))^{-1} f_\omega \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} n g \right) \mathrm{d}n = \sum_{\gamma \in \Gamma \, : \, \gamma \mathfrak{b} = \mathfrak{a}} \int\limits_{B^+ \backslash N} (\psi_{\omega'}(n))^{-1} \psi_{(\eta(\gamma))^2 \omega}(n) \, \mathrm{d}n \, f_\omega \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} g \right) = \\ = \sum_{\gamma \in \Gamma \, : \, \gamma \mathfrak{b} = \mathfrak{a}} \delta_{\omega', (\eta(\gamma))^2 \omega} \, f_\omega \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} g \right) . \tag{6.2.16}$$

As for the second sum on the right-hand side of (6.2.15), it follows by (1.5.8) that if $0 \neq c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ then, for each $\gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)$, there is some $(s, d) = (s(\gamma), d(\gamma)) \in \mathbb{C}^2$ such that

$$g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} = n[s/c] h[1/c] k[0, -1] n[d/c] \tag{6.2.17}$$

(this being essentially the same device introduced in the equation (6.1.8) of Lemma 6.1.1). Hence, supposing now that $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ is given, it follows (by the hypothesis that $f_\omega \in C^0(N \backslash G, \omega)$, and the fact that $\mathrm{d}n$ is a Haar measure for N) that if $\gamma \in {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c)$ then

$$\int\limits_N (\psi_{\omega'}(n))^{-1} f_\omega \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} n g \right) \mathrm{d}n = \int\limits_N (\psi_{\omega'}(n))^{-1} f_\omega \left(n[s(\gamma)/c] h[1/c] k[0, -1] n[d(\gamma)/c] n g \right) \mathrm{d}n = \\ = \psi_\omega \left(n[s(\gamma)/c] \right) \psi_{\omega'} \left(n[d(\gamma)/c] \right) \int\limits_N (\psi_{\omega'}(n))^{-1} f_\omega \left(h[1/c] k[0, -1] n g \right) \mathrm{d}n \, .$$

By (1.5.2) and (1.5.7), the last integral in the above may be expressed as the Jacquet integral $(\mathbf{J}_{\omega'} \mathbf{h}_{1/c} f_\omega)(g)$; since this integral is independent of the 'γ' in the above equations ('c' being fixed there), and since

$$\sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c) / \Gamma'_{\mathfrak{b}}} \psi_\omega \left(n[s(\gamma)/c] \right) \psi_{\omega'} \left(n[d(\gamma)/c] \right) = S_{\mathfrak{a}, \mathfrak{b}} \left(\omega, \omega'; c \right)$$

(see (1.4.3), (1.5.10) and (6.2.17)), it therefore follows that, for $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$,

$$\sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(c) / \Gamma'_{\mathfrak{b}}} \int\limits_N (\psi_{\omega'}(n))^{-1} f_\omega \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} n g \right) \mathrm{d}n = S_{\mathfrak{a}, \mathfrak{b}} \left(\omega, \omega'; c \right) \left(\mathbf{J}_{\omega'} \mathbf{h}_{1/c} f_\omega \right)(g) \, .$$

By combining this last result with (6.2.15) and (6.2.16), we arrive at the case $\mathfrak{a}' = \mathfrak{b}$ of the formula stated in (1.5.5). By consideration of the steps used to obtain this formula, it may be seen that the absolute convergence of all the sums and integrals on the right-hand side of (1.5.5) is a direct consequence of Lemma 6.2.2 (this becomes clear if one lets $|f_\omega|$ be substituted for f_ω, and then considers the case $\omega' = \omega = 0$) ∎

Remark 6.2.6. The Fourier expansion (1.8.4)-(1.8.6) of the Eisenstein series $E^{\mathfrak{a}}_{\ell,q}(\nu,p)$ may be shown to follow, via (1.4.1)-(1.4.3), from the case $\omega=0$, $f=f_0=\varphi_{\ell,k}(\nu,p)$ of Lemma 6.2.5: note in particular that, by the definitions in (1.3.2) and (6.2.1), the hypothesis that $\mathrm{Re}(\nu)>1$ suffices to ensure that the condition (6.2.5) is satisfied when $f_\omega=\varphi_{\ell,q}(\nu,p)$ and $R_0=1$ (say). The use of the Fourier expansion (1.4.1) in this context may be justified through an appeal to Theorem 67 of [2]: we skip the relevant details, which are similar to what occurs in the last two paragraphs of the proof of Lemma 6.5.14, below.

By making use of the meromorphic continuation of the functions $\nu\mapsto E^{\mathfrak{a}}_{\ell,q}(\nu,p)$ and $\nu\mapsto D^{\mathfrak{b}}_{\mathfrak{a}}(\psi;\nu,p)$, which is discussed in Subsection 1.8, one may dispense with the condition $\mathrm{Re}(\nu)>1$; the Fourier expansion in (1.8.4) is thereby obtained for all $(\nu,p)\in(\mathbb{C}\times\mathbb{Z})-\{(0,0),(1,0)\}$ such that $\mathrm{Re}(\nu)\geq 0$ and $|p|\leq\ell$.

For the proof of the spectral sum formula, Theorem B, we require certain 'cusp sector estimates' for the Eisenstein series $E^{\mathfrak{a}}_{\ell,q}(\nu,p)$ and other Poincaré series; we deduce the required estimates (those of Lemma 6.2.8 and Lemma 6.2.9, below) with the help of the following lemma, which is taken from Section 5.2 of [32].

Lemma 6.2.7. *Let $\ell,p,q\in\mathbb{Z}$ satisfy $\ell\geq\max\{|p|,|q|\}$; let $\nu\in\mathbb{C}$; let $\mathfrak{b}$ be a cusp of Γ; and let $g_{\mathfrak{b}}\in G$ be such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c}=\mathfrak{b}$. Suppose moreover that $f:G\to\mathbb{C}$ satisfies*

$$f\left(g_{\mathfrak{b}}g\right)=\sum_{0\neq\omega\in\mathfrak{O}}c(\omega)\left(\mathbf{J}_\omega\varphi_{\ell,q}(\nu,p)\right)(g)\qquad(g\in G),\tag{6.2.18}$$

where, for $0\neq\omega\in\mathfrak{O}$, the coefficient $c(\omega)\in\mathbb{C}$ is independent of g. Then there exists some $r_0\in[1,\infty)$ such that

$$\left|f\left(g_{\mathfrak{b}}g\right)\right|\leq e^{-\pi\rho(g)}\qquad\text{for all } g\in G \text{ with } \rho(g)\geq r_0.\tag{6.2.19}$$

Proof. This lemma is a slight variation on the special case of Part (i) of Lemma 5.2.1 of [32] in which one has, in the notation of [32], $p\in\mathbb{Z}$ (rather than $2p\in\mathbb{Z}-2\mathbb{Z}$), $\Lambda'_\kappa=(1/2)\mathfrak{O}$ and $\chi:\Gamma\to\{1\}$. Note, in particular, that it is clear from the proof given in [32] that the upper bounds stated in the equations (5.9) and (5.10) of [32] are not optimal, and may indeed be sharpened by any factor of the form $\exp(-(1-\varepsilon)2\pi\omega_0 r)$, where ε denotes an arbitrarily small positive absolute constant, while $\omega_0=\min\{|\omega|:0\neq\omega\in\Lambda'_\kappa\}$ (so that $\omega_0=1/2$ when $\Lambda'_\kappa=(1/2)\mathfrak{O}$). Hence, by assuming greater lower bounds for $r=\rho(g)$ than those implicit in the equation (5.9) of [32], one may sharpen the upper bound stated there by any factor of the form $O(r^{\ell+1/2})$ ■

Lemma 6.2.8. *Let $\ell,q\in\mathbb{Z}$ satisfy $\ell\geq|q|$, let $(\nu,p)\in\mathbb{C}\times\mathbb{Z}-\{(0,0),(1,0)\}$ be such that $\mathrm{Re}(\nu)\geq 0$ and $|p|\leq\ell$; let $\mathfrak{a},\mathfrak{b}$ be cusps of Γ, and let $g_{\mathfrak{a}},g_{\mathfrak{b}}\in G$ be such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c}\in\{\mathfrak{a},\mathfrak{b}\}$. Then, for some $r_0=r_0(\Gamma,\ell,\nu)\in[1,\infty)$, and some $\epsilon=\epsilon(g_{\mathfrak{a}},g_{\mathfrak{b}})\in\mathfrak{O}^*$ satisfying $\epsilon=1$ if $\mathfrak{a}=\mathfrak{b}$ and $g_{\mathfrak{a}}=g_{\mathfrak{b}}$, one has:*

$$\epsilon^p E^{\mathfrak{a}}_{\ell,q}(\nu,p)\left(g_{\mathfrak{b}}g\right)=\delta^{\Gamma}_{\mathfrak{a},\mathfrak{b}}\varphi_{\ell,q}(\nu,p)(g)+O_{\Gamma,\ell,\nu}\left((\rho(g))^{1-\mathrm{Re}(\nu)}\right)\qquad\text{for all } g\in G \text{ with } \rho(g)\geq r_0.\tag{6.2.20}$$

Proof. We may suppose that either $\mathfrak{a}=\mathfrak{b}$ and $g_{\mathfrak{a}}=g_{\mathfrak{b}}$, or else the cusps $\mathfrak{a}$ and $\mathfrak{b}$ are not Γ-equivalent (by virtue of the point noted two lines below (1.8.3), these two cases of the lemma imply every other case). Then, in light of Remark 6.2.6, we have the Fourier expansion of $E^{\mathfrak{a}}_{\ell,q}(\nu,p)$ shown in equation (1.8.4). It follows that by putting, for $g\in G$,

$$f(g)=E^{\mathfrak{a}}_{\ell,q}(\nu,p)(g)-\delta^{\Gamma}_{\mathfrak{a},\mathfrak{b}}\varphi_{\ell,q}(\nu,p)\left(g_{\mathfrak{b}}^{-1}g\right)-\frac{D^{\mathfrak{b}}_{\mathfrak{a}}(0;\nu,p)}{[\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]}\,\frac{\pi\Gamma(|p|+\nu)}{\Gamma(\ell+1+\nu)}\,\frac{\Gamma(\ell+1-\nu)}{\Gamma(|p|+1-\nu)}\,\varphi_{\ell,q}(-\nu,-p)\left(g_{\mathfrak{b}}^{-1}g\right)$$

we define a function $f:G\to\mathbb{C}$ satisfying, for a certain choice of coefficients $c(\omega)$ $(\omega\in\mathfrak{O}-\{0\})$, the hypothesis (6.2.18) of Lemma 6.2.7; the result (6.2.19) of that lemma therefore implies that there exists some $r_0\in[1,\infty)$ such that, for all $g\in G$ with $\rho(g)\geq r_0$, one has the equation

$$E^{\mathfrak{a}}_{\ell,q}(\nu,p)\left(g_{\mathfrak{b}}g\right)=\delta^{\Gamma}_{\mathfrak{a},\mathfrak{b}}\varphi_{\ell,q}(\nu,p)(g)+O_{\Gamma,g_{\mathfrak{a}},g_{\mathfrak{b}},\ell,q,p,\nu}\left((\rho(g))^{1-\mathrm{Re}(\nu)}\right)+O\left(e^{-\pi\rho(g)}\right)\tag{6.2.21}$$

(in which the first of the two O-terms represents an estimate for the third term on the right-hand side of the equation defining $f(g)$, and is justified by virtue of the definitions (1.3.2), (1.5.10), (1.8.6), (6.2.1) and the case $\psi = 0$ of the meromorphic continuation of $D_{\mathfrak{a}}^{\mathfrak{b}}(\psi; \nu, p)$ discussed in Subsection 1.8).

Since $r_0 \geq 1$, the final O-term in (6.2.21) may be omitted: for one has $e^{-\pi\rho} = O_\nu(\rho^{1-\mathrm{Re}(\nu)})$ when $\rho \geq 1$. Moreover, since p and q must lie in the finite set $\{-\ell, -\ell+1, \ldots, \ell\}$, the implicit constant associated with the first O-term of (6.2.21) may be chosen so as to depend only upon Γ, $g_{\mathfrak{a}}$, $g_{\mathfrak{b}}$, ℓ and ν; indeed, by the point noted two lines below equation (1.8.3), and by the fact that the number of Γ-equivalence classes of cusps is finite, it follows that this implicit constant need depend only upon Γ, ℓ and ν; for the same reasons, a suitable choice of r_0 (in the above) may be determined from just Γ, ℓ and ν ■

Square integrable Poincaré series. We assume (as in the preceding discussion) that $\mathfrak{a}$ is a cusp of Γ, that $\omega \in \mathfrak{O}$, and that, for some $\sigma_0 > 1$, and some $R_0 > 0$, the function $f_\omega \in C^0(N\backslash G, \omega)$ satisfies the condition (6.2.5). In order to establish the square-integrability (over $\Gamma\backslash G$) of the Poincaré series $P^{\mathfrak{a}} f_\omega$, we require also that, for some $\sigma_\infty > 0$, and some $R_\infty \in (0, \infty)$, the function f_ω satisfies:

$$f_\omega(g) \ll_{f_\omega} (\rho(g))^{1-\sigma_\infty} \qquad \text{for all } g \in G \text{ such that } \rho(g) \geq R_\infty. \tag{6.2.22}$$

This implies (just as (6.2.5) implies (6.2.6)) that if $R > 0$ then

$$f_\omega(g) \ll_{f_\omega, \sigma_\infty, R_\infty, R} (\rho(g))^{1-\sigma_\infty} \qquad \text{for all } g \in G \text{ such that } \rho(g) \geq R. \tag{6.2.23}$$

Lemma 6.2.9. *Let $0 \neq q_0 \in \mathfrak{O}$ and $\Gamma = \Gamma_0(q_0) \leq SL(2, \mathfrak{O})$; let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of Γ; let $g_{\mathfrak{a}}, g_{\mathfrak{b}} \in G$ be such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} \in \{\mathfrak{a}, \mathfrak{b}\}$; let $m_{\mathfrak{b}} \in \mathfrak{O} - \{0\}$ be as described below (1.1.22); let $\omega \in \mathfrak{O}$, $\sigma_0 > 1$, $R_0 > 0$, $\sigma_\infty > 0$ and $R_\infty \in (0, \infty)$, and let $f = f_\omega \in C^0(N\backslash G, \omega)$ be such that both of the conditions (6.2.5) and (6.2.22) are satisfied. Then, for all $g \in G$ such that $\rho(g) > 1/|m_{\mathfrak{b}}|$, one has:*

$$\left(P^{\mathfrak{a}} f_\omega\right)(g_{\mathfrak{b}} g) = \delta_{\mathfrak{a},\mathfrak{b}}^{\Gamma}\, O_{\Gamma, f, \sigma_\infty, R_\infty}\left((\rho(g))^{1-\sigma_\infty}\right) + O_{\Gamma, f, \sigma_0, R_0}\left((\rho(g))^{1-\sigma_0}\right). \tag{6.2.24}$$

Proof. It will suffice to prove that (6.2.24) holds if if $\mathfrak{a} = \mathfrak{b}$ and $g_{\mathfrak{a}} = g_{\mathfrak{b}}$, or if the cusps $\mathfrak{a}$ and $\mathfrak{b}$ are not Γ-equivalent: for in every other case one has $\mathfrak{a} = \tau\mathfrak{b}$ for some $\tau \in \Gamma$, which (by (1.1.4), (6.2.1) and the result (2.1) of Lemma 2.1, and by virtue the fact that $P^{\mathfrak{a}} f_\omega$ is Γ-automorphic) implies that, for $g \in G$, one has $(P^{\mathfrak{a}} f_\omega)(g_{\mathfrak{b}} g) = (P^{\mathfrak{a}} f_\omega)(\tau g_{\mathfrak{b}} g) = (P^{\mathfrak{a}} f_\omega)(g_{\mathfrak{a}} \tilde{g})$, where $\tilde{g} = g_{\mathfrak{a}}^{-1} \tau g_{\mathfrak{b}} g$ is such that $\rho(\tilde{g}) = \rho(g)$.

In cases where $\mathfrak{a}$ and $\mathfrak{b}$ are not Γ-equivalent the set ${}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0)$ (defined in (1.5.8)) is empty, and so in these cases it follows by the result (6.2.4) of Lemma 6.2.1 that if $g \in G$, and if $\rho(g) > 1/|m_{\mathfrak{b}}|$, then

$$\rho\left(g_{\mathfrak{a}} \gamma g_{\mathfrak{b}} g\right) < 1 \qquad \text{for all } \gamma \in \Gamma. \tag{6.2.25}$$

Therefore one may, in such cases, apply the definition (1.5.4) and hypothesis (6.2.5) (with its corollary (6.2.6)) so as to obtain, for $g \in G$ such that $\rho(g) > 1/|m_{\mathfrak{b}}|$,

$$\begin{aligned}\left(P^{\mathfrak{a}} f_\omega\right)(g_{\mathfrak{b}} g) &= \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} g\right) = \\ &= \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma} O_{f_\omega, \sigma_0, R_0}\left(\left(\rho\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} g\right)\right)^{1+\sigma_0}\right) \ll_{f_\omega, \sigma_0, R_0} E_{0,0}^{\mathfrak{a}}(\sigma_0, 0)(g_{\mathfrak{b}} g)\end{aligned} \tag{6.2.26}$$

(the final upper bound here following by (6.2.2) and (1.8.1)). By (6.2.26) and the result (6.2.20) of Lemma 6.2.8, we obtain proof of those cases of the lemma in which $\mathfrak{a}$ and $\mathfrak{b}$ are not Γ-equivalent.

Given the conclusion just reached, and point noted at the beginning of this proof, we may assume henceforth that $\mathfrak{a} = \mathfrak{b}$ and $g_{\mathfrak{a}} = g_{\mathfrak{b}}$. Consequently we have only to obtain a suitable bound for $(P^{\mathfrak{a}} f_\omega)(g_{\mathfrak{a}} g)$, and (in doing so) may assume that $g \in G$ is such that $\rho(g) > 1/|m_{\mathfrak{a}}|$.

By the definition (1.5.4) and Lemma 6.2.1, and by (1.5.8) and (1.1.16), we find that

$$\left(P^{\mathfrak{a}} f_\omega\right)(g_{\mathfrak{a}} g) = S_0(g) + S_1(g) \ , \tag{6.2.27}$$

where

$$S_0(g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{a}}(0)} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g\right) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma_{\mathfrak{a}}} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g\right)$$

and

$$S_1(g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{0 \neq c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{a}}(c)} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g\right) .$$

As a consequence of both the result (6.2.4) of Lemma 6.2.1 and the hypothesis (6.2.5) (with corollary (6.2.6)), it follows (similarly to how (6.2.25) and (6.2.26) were obtained) that one has

$$\begin{aligned} S_1(g) &\ll_{f_\omega, \sigma_0, R_0} \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{0 \neq c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{a}}} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash {}^{\mathfrak{a}}\Gamma^{\mathfrak{a}}(c)} \left(\rho\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g\right)\right)^{1+\sigma_0} = \\ &= E^{\mathfrak{a}}_{0,0}(\sigma_0, 0)(g_{\mathfrak{a}} g) - \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma_{\mathfrak{a}}} \varphi_{0,0}(\sigma_0, 0)\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g\right) . \end{aligned}$$

By Lemma 4.2 and (1.8.2) (or (6.2.1), (6.2.2) and (1.1.4)), we have here $\varphi_{0,0}(\sigma_0, 0)(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g) = \varphi_{0,0}(\sigma_0, 0)(g)$ when $\gamma \in \Gamma_{\mathfrak{a}}$, and therefore may deduce (using Lemma 6.2.8) that

$$S_1(g) = O_{f_\omega, \sigma_0, R_0}\left(E^{\mathfrak{a}}_{0,0}(\sigma_0, 0)(g_{\mathfrak{a}} g) - \varphi_{0,0}(\sigma_0, 0)(g)\right) = O_{f_\omega, \sigma_0, R_0, \Gamma}\left((\rho(g))^{1-\sigma_0}\right) . \tag{6.2.28}$$

We now consider the sum $S_0(g)$, defined below (6.2.27). By the result (6.2.3) of Lemma 6.2.1, we have $\rho(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g) = \rho(g)$ for all $\gamma \in \Gamma_{\mathfrak{a}} = {}^{\mathfrak{a}}\Gamma^{\mathfrak{a}}(0)$. Therefore, given that $\rho(g) > 1/|m_{\mathfrak{b}}| \geq 1/|q_0|$ (the last inequality following by (6.1.25)), it is a consequence of the hypothesis (6.2.22) (with corollary (6.2.23)) that we have

$$S_0(g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma_{\mathfrak{a}}} O_{f_\omega, \sigma_\infty, R_\infty, \Gamma}\left(\left(\rho\left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{a}} g\right)\right)^{1-\sigma_\infty}\right) \ll_{f_\omega, \sigma_\infty, R_\infty, \Gamma} (\rho(g))^{1-\sigma_\infty} .$$

By this last bound, that in (6.2.28), and the equation (6.2.27), we obtain the case $\mathfrak{a} = \mathfrak{b}$, $g_{\mathfrak{a}} = g_{\mathfrak{b}}$ of the lemma, and so complete its proof ■

Corollary 6.2.10. *Let those of the hypotheses of the above lemma that concern q_0, Γ, $\mathfrak{a}$, $g_{\mathfrak{a}}$, ω, σ_0, R_0, σ_∞, R_∞ and f_ω be satisfied. Then one has $P^{\mathfrak{a}} f_\omega \in L^2(\Gamma \backslash G)$; if, moreover, $\sigma_\infty \geq 1$ then $P^{\mathfrak{a}} f_\omega$ is bounded on G.*

Proof. It may be proved, similarly to Proposition 2.2.4 of [11], that the inner product $\langle f, g \rangle_{\Gamma \backslash G}$ defined in (1.2.2) is independent of the choice of fundamental domain for the action of Γ upon $\mathbb{H}_3$. Hence, by choosing to replace $\mathcal{F}$ (in (1.2.2)) by a fundamental domain $\mathcal{F}_*$ fitting the description given in (1.1.22)-(1.1.24), we find that

$$\|P^{\mathfrak{a}} f_\omega\|^2_{\Gamma \backslash G} = \langle P^{\mathfrak{a}} f_\omega, P^{\mathfrak{a}} f_\omega \rangle_{\Gamma \backslash G} = \int_{\mathcal{F}_*} \int_{K^+} \left|\left(P^{\mathfrak{a}} f_\omega\right)(n[z] a[r] k)\right|^2 \mathrm{d}k \, r^{-3} d_+ z \, \mathrm{d}r$$

(provided that the last integral exists). Therefore, and since the union formed in (1.1.24) is (see Lemma 2.2) a union of a finite number of sets, we have $P^{\mathfrak{a}} f_\omega \in L^2(\Gamma \backslash G)$ if, when $\mathfrak{C}(\Gamma)$, $\mathcal{D}$ and the family of sets $(\mathcal{E}_{\mathfrak{c}})_{\mathfrak{c} \in \mathfrak{C}(\Gamma)}$ are as described in the paragraph containing (1.1.23)-(1.1.24), one has:

$$\int_{\mathcal{X}} \int_K \left|\left(P^{\mathfrak{a}} f_\omega\right)(n[z] a[r] k)\right|^2 \mathrm{d}k \, r^{-3} d_+ z \, \mathrm{d}r < \infty \qquad \text{for } \mathcal{X} \in \{\mathcal{D}\} \cup \{\mathcal{E}_{\mathfrak{c}} : \mathfrak{c} \in \mathfrak{C}(\Gamma)\}. \tag{6.2.29}$$

We consider firstly the case $\mathcal{X} = \mathcal{D}$ (some compact hyperbolic polygon, contained in $\mathbb{H}_3$). By the Iwasawa decomposition of G, one has

$$\int_{\mathcal{D}} \int_K \left|(P^{\mathfrak{a}} f_\omega)\,(n[z]a[r]k)\right|^2 \mathrm{d}k\, r^{-3} d_+ z\, \mathrm{d}r = \int_{\tilde{\mathcal{D}}} \Phi(g)\, \mathrm{d}g\ , \tag{6.2.30}$$

where $\tilde{\mathcal{D}} = \{n[z]a[r] : (z,r) \in \mathcal{D}\}K \subset G$ and $\Phi : \tilde{\mathcal{D}} \to [0,\infty)$ is given by $\Phi(g) = |(P^{\mathfrak{a}} f_\omega)(g)|^2$ (for $g \in \tilde{\mathcal{D}}$). The mapping $(z,r) \mapsto n[z]a[r]$ is a homeomorphism from the compact subset $\mathcal{D}$ of $\mathbb{H}_3$ onto the subset $\{n[z]a[r] : (z,r) \in \mathcal{D}\}$ of G, and so the latter set is a compact subset of G; since the subgroup $K = SU(2)$ of G is also compact, one can deduce that the set $\tilde{\mathcal{D}} = \{n[z]a[r] : (z,r) \in \mathcal{D}\}K$ has the Bolzano-Weierstrass property and is, therefore, a compact subset of G. Corollary 6.2.3 implies that Φ is continuous on $\tilde{\mathcal{D}}$. By the compactness of $\tilde{\mathcal{D}}$ and the continuity of Φ, it follows that we have $\int_{\tilde{\mathcal{D}}} \Phi(g)\, \mathrm{d}g < \infty$ in the equation (6.2.30). This proves the case $\mathcal{X} = \mathcal{D}$ of (6.2.29).

Suppose now that $\mathfrak{c} \in \mathfrak{C}(\Gamma)$. Then, by (1.1.23), the definition (above (1.1.4)) of the action of G on $\mathbb{H}_3$, and the fact that the measure $\mathrm{d}g$ defined in (1.1.11) is a Haar measure, one has:

$$\int_{\mathcal{E}_{\mathfrak{c}}} \int_K \left|(P^{\mathfrak{a}} f_\omega)\,(n[z]a[r]k)\right|^2 \mathrm{d}k\, r^{-3} d_+ z\, \mathrm{d}r = \int_{\mathcal{R}_{\mathfrak{c}}} \int_K \int\limits_{1/|m_{\mathfrak{c}}|}^{\infty} \left|(P^{\mathfrak{a}} f_\omega)\,(g_{\mathfrak{c}} n[z]a[r]k)\right|^2 r^{-3}\, \mathrm{d}r\, \mathrm{d}k\, d_+ z \tag{6.2.31}$$

(this equation being justified by the Tonelli-Hobson Test for integrabililty, Theorem 15.8 of [1], provided that the iterated integral on the right-hand side exists). Now, by (1.1.10) and (1.1.22), one has

$$\int_{\mathcal{R}_{\mathfrak{c}}} \int_K \int\limits_{1/|m_{\mathfrak{c}}|}^{\infty} \left(r^{1-\sigma}\right)^2 r^{-3}\, \mathrm{d}r\, \mathrm{d}k\, d_+ z = \frac{4}{[\Gamma_{\mathfrak{c}} : \Gamma_{\mathfrak{c}}']} \int\limits_{1/|m_{\mathfrak{c}}|}^{\infty} r^{-(1+2\sigma)}\, \mathrm{d}r < \infty \qquad \text{for } \sigma > 0.$$

Therefore, given the definition (6.2.1) and Corollary 6.2.3, and given that we have both $\sigma_0 > 0$ and $\sigma_\infty > 0$, it follows from (6.2.31), by an application of the result (6.2.24) of Lemma 6.2.9, that the case $\mathcal{X} = \mathcal{E}_{\mathfrak{c}}$ of (6.2.29) holds. This completes the proof that the function $P^{\mathfrak{a}} f_\omega$ lies in the space $L^2(\Gamma\backslash G)$.

To obtain the final result of the corollary we first note that $P^{\mathfrak{a}} f_\omega$ is a continuous function on G, and so is bounded on the compact set $\tilde{\mathcal{D}} = \{n[z]a[r] : (z,r) \in \mathcal{D}\}K \subset G$. Moreover, if $\sigma_\infty \geq 1$ then it is implied by Lemma 6.2.9 that, for $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, the function $P^{\mathfrak{a}} f_\omega$ is bounded on the set $\{n[z]a[r] : (z,r) \in \mathcal{E}_{\mathfrak{c}}\}K \subset G$. Given these facts, and given the definition of $\mathcal{F}_*$ in (1.1.24), we may conclude that in cases where $\sigma_\infty \geq 1$ the function $P^{\mathfrak{a}} f_\omega$ is bounded on the set $\tilde{\mathcal{F}}_* = \{n[z]a[r] : (z,r) \in \mathcal{F}_*\}K^+ \subset G$; since this set $\tilde{\mathcal{F}}_*$ is a fundamental domain for $\Gamma\backslash G$, and since the function $P^{\mathfrak{a}} f_\omega : G \to \mathbb{C}$ is Γ-automorphic, it therefore follows that $P^{\mathfrak{a}} f_\omega$ is bounded on G if $\sigma_\infty \geq 1$ $\blacksquare$

§6.3 The Goodman-Wallach operator $\mathbf{M}_\omega$.

Let $\nu \in \mathbb{C}$ and $p \in \mathbb{Z}$. Bruggeman and Motohashi, in Section 6 of [5], employ a method of Goodman and Wallach [12] in constructing a certain family of linear operators $(\mathbf{M}_\omega^{\nu,p})_{\omega\in\mathbb{C}}$, with common domain $H(\nu,p)$. In this subsection we detail the salient properties of these operators; we omit the relevant proofs, which may be found in our sources, [5] and [32].

Let $\omega \in \mathbb{C}$. Then, for $\varphi \in H(\nu,p)$, Bruggeman and Motohashi define $\mathbf{M}_\omega \varphi = \mathbf{M}_\omega^{\nu,p} \varphi$ by putting

$$\left(\mathbf{M}_\omega \varphi\right)(g) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_\omega(\nu,p;m,n) \left(\frac{\partial}{\partial z}\right)^m \left(\frac{\partial}{\partial \overline{z}}\right)^n \varphi\big(k[0,-1]n[z](k[0,-1])^{-1}g\big)\Big|_{z=0} \qquad (g \in G), \tag{6.3.1}$$

where

$$a_\omega(\nu,p;m,n) = \frac{(\pi i \omega)^m\, (\pi i\, \overline{\omega})^n}{(m!)(n!)\Gamma(\nu+1-p+m)\Gamma(\nu+1+p+n)}\ . \tag{6.3.2}$$

The double series in (6.3.1) is absolutely convergent: this follows (as observed in Section 6 of [5]) by virtue of the function $(x,y) \mapsto \varphi(k[0,-1]n[x+iy]k[0,-1]^{-1}g)$ being real analytic.

The operator $\mathbf{M}_\omega = \mathbf{M}_\omega^{\nu,p}$ commutes with all elements of $\mathcal{U}(\mathfrak{g})$, and satisfies

$$(\mathbf{M}_\omega \varphi)(ng) = \psi_\omega(n)(\mathbf{M}_\omega \varphi)(g) \qquad \text{for } \varphi \in H(\nu,p),\ g \in G \text{ and } n \in N \tag{6.3.3}$$

(see Section 6 of [5], where the correspondence between (6.3.2) and (6.3.3) is established). Therefore one has $\mathbf{M}_\omega \varphi_{\ell,q}(\nu,p) \in W_\omega(\Upsilon_{\nu,p};\ell,q)$ for $\ell, q \in \mathbb{Z}$ with $\ell \geq |p|$ and $|q| \leq \ell$, so that

$$\mathbf{M}_\omega : H(\nu,p) \to \bigoplus_{\ell=|p|}^{\infty} \bigoplus_{q=-\ell}^{\ell} W_\omega(\Upsilon_{\nu,p};\ell,q). \tag{6.3.4}$$

We assume, for the remainder of this subsection, that ℓ and q are integers satisfying $\ell \geq \max\{|p|,|q|\}$. Since $(h[u])^{-1}k[0,-1]n[z](k[0,-1])^{-1}h[u] = k[0,-1]n[u^2 z](k[0,-1])^{-1}$, it follows directly from (6.3.1) that for $u \in \mathbb{C}^*$ one has $\mathbf{h}_u \mathbf{M}_\omega \mathbf{h}_u^{-1} = \mathbf{M}_{u^2\omega}$ (with $\mathbf{h}_u$ as defined in (1.5.7)). By this and (1.8.2), one finds that

$$\mathbf{h}_u \mathbf{M}_\omega \varphi_{\ell,q}(\nu,p) = |u|^{2(1+\nu)}(u/|u|)^{-2p}\mathbf{M}_{u^2\omega}\varphi_{\ell,q}(\nu,p) \qquad (u \in \mathbb{C}^*). \tag{6.3.5}$$

In Lemma 6.1 of [5] it is shown that if $\omega \neq 0$ then one may express $(\mathbf{M}_\omega \varphi_{\ell,q}(\nu,p))(na[r]k)$, for $n \in N$, $r > 0$ and $k \in K$, as a sum of finitely many terms of the form $c\psi_\omega(n) r^{a+1} I_{\nu+a-b}(2\pi|\omega|r)\Phi_{m,q}^{\ell}(k)$, where c is a constant and a, b and m are integers satisfying $a, b \geq 0$ and $|m| \leq \ell$, while $I_\mu(z)$ is the 'modified' Bessel function $i^{-\mu}J_\mu(iz)$ (with $J_\mu(w)$ as in (1.9.6)-(1.9.8)) and $\Phi_{m,q}^{\ell}(k)$ is as in (1.3.2). Since $I_\mu(y) \sim (2\pi y)^{-1/2}e^y$ as $y \to +\infty$ (with $\mu \in \mathbb{C}$ a constant, and $y \in \mathbb{R}$), it follows from Lemma 6.1 of [5] that if $\omega \neq 0$ then the function $r \mapsto |(\mathbf{M}_\omega \varphi_{\ell,q}(\nu,p))(na[r]k)|$ is exponentially increasing as $r \to +\infty$. This implies (given (1.5.16), (1.4.9) and (1.4.11)) that for $\omega \neq 0$ the Jacquet integral $\mathbf{J}_\omega \varphi_{\ell,q}(\nu,p)$ and 'Goodman-Wallach transform' $\mathbf{M}_\omega \varphi_{\ell,q}(\nu,p)$ are two linearly independent functions that together span the space $W_\omega(\Upsilon_{\nu,p};\ell,q)$. The proof of Lemma 6.1 of [5] shows moreover that if $\omega \neq 0$ and $(\nu,p) \neq (0,0)$ then another basis for $W_\omega(\Upsilon_{\nu,p};\ell,q)$ is $\{\mathbf{M}_\omega \varphi_{\ell,q}(\nu,p), \mathbf{M}_\omega \varphi_{\ell,q}(-\nu,-p)\}$. One has, in particular, the following relation of linear dependence

$$\begin{aligned}(\pi|\omega|)^{-\nu}\left(\frac{-i\omega}{|\omega|}\right)^{p}\Gamma(\ell+1+\nu)\mathbf{J}_\omega\varphi_{\ell,q}(\nu,p) &= \\ = \sum_{(\mu,\varpi)=\pm(\nu,p)} \frac{\pi^2(\pi|\omega|)^{\mu}}{\sin(-\pi\mu)}\left(\frac{i\omega}{|\omega|}\right)^{-\varpi}&\Gamma(\ell+1+\mu)\mathbf{M}_\omega\varphi_{\ell,q}(\mu,\varpi)\,,\end{aligned} \tag{6.3.6}$$

which is Equation (6.15) of Lemma 6.1 of [5] (and which is, moreover, valid whenever $\omega \neq 0$, provided that in cases where ν is an integer one defines both sides of the equation via the relevant analytic continuation).

The next lemma supplies a collection of useful estimates for Goodman-Wallach transforms on subsets of G of the form $\{a[r]k : 0 < r \leq r_1, k \in K\}$, with r_1 small and positive; by (6.3.3), each such estimate implies an equally strong estimate on the larger set $\{na[r]k : n \in N, 0 < r \leq r_1, k \in K\}$.

Lemma 6.3.1 (Bruggeman and Motohashi). *Let $\omega \neq 0$ and $r_1 \in (0,\infty)$. Then one has*

$$(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,p))(a[r]k) = \frac{r^{1+\nu}}{\Gamma(\nu+1-p)\Gamma(\nu+1+p)}\Phi_{p,q}^{\ell}(k) + O\left(r^{2+\mathrm{Re}(\nu)}\right), \tag{6.3.7}$$

$$(\mathbf{M}_\omega\varphi_{\ell,q}(0,p))(a[r]k) = \frac{1}{|p|!}\begin{pmatrix}\ell \\ \ell-|p|\end{pmatrix}(\pi|\omega|)^{|p|}\left(\frac{-i\omega}{|\omega|}\right)^{p} r^{1+|p|}\Phi_{0,q}^{\ell}(k) + O\left(r^{2+|p|}\right), \tag{6.3.8}$$

uniformly for $r \in (0,r_1]$ and $k \in K$ (the implicit constant in (6.3.7) depends only on ω, ℓ, ν and r_1; that in (6.3.8) depends only on ω, ℓ and r_1). If, moreover, $\sigma_0 \in (0,\infty)$ then one has also

$$(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,p))(a[r]k) \ll r^{1+\mathrm{Re}(\nu)}(1+|\mathrm{Im}(\nu)|)^{-2\mathrm{Re}(\nu)-1}e^{\pi|\mathrm{Im}(\nu)|}, \tag{6.3.9}$$

uniformly for $0 < r \leq r_1$, $k \in K$ and $|\mathrm{Re}(\nu)| \leq \sigma_0$ (the implicit constant depends only on ω, ℓ, r_1 and σ_0).

Proof. The results (6.3.7)-(6.3.8) follow from the equations (6.13) and (6.14) of Lemma 6.1 of [5], the power-series expansion $I_\mu(z) = \sum_{m\geq 0}(m!\Gamma(\mu+1+m))^{-1}(z/2)^{\mu+2m}$ and identities $I_{-n}(z) = I_n(z)$ $(n \in \mathbb{Z})$.

The calculation (7.15) of [5] makes implicit use of (6.3.8) and the case $\nu = 1$, $p = 0$ of (6.3.7). Equation (6.3.7) and the bound (6.3.9) are the results (4.53) and (4.55) of [32]. The result (6.3.9) is a corollary of the upper bound for $|I_\mu(z)|$ stated in Relation (1.32) of [32] ■

The next lemma is used in the proof of Lemma 6.5.10, where has a part to play in the calculation of the Fourier expansion of the Poincaré series $P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$.

Lemma 6.3.2 (Bruggeman and Motohashi). *Let $0 \neq \omega_1 \in \mathbb{C}$ and $\mathrm{Re}(\nu) > 0$. Then*

$$\mathbf{J}_0\mathbf{M}_{\omega_1}\varphi_{\ell,q}(\nu,p) = (-1)^p \,\frac{\sin(\pi\nu)}{(\nu^2 - p^2)}\,\frac{\Gamma(\ell+1-\nu)}{\Gamma(\ell+1+\nu)}\,\varphi_{\ell,q}(-\nu,-p) \tag{6.3.10}$$

and, for $\omega_2 \in \mathbb{C}$ with $\omega_2 \neq 0$, one has

$$\mathbf{J}_{\omega_2}\mathbf{M}_{\omega_1}\varphi_{\ell,q}(\nu,p) = \mathcal{J}^*_{\nu,p}\left(2\pi\sqrt{\omega_1\omega_2}\right)\mathbf{J}_{\omega_2}\varphi_{\ell,q}(\nu,p)\ , \tag{6.3.11}$$

where, in terms of the notation defined in (1.9.5)-(1.9.6) (Theorem B),

$$\mathcal{J}^*_{\nu,p}(z) = |z/2|^{-2\nu}(z/|z|)^{2p}\mathcal{J}_{\nu,p}(z) = J^*_{\nu-p}(z)J^*_{\nu+p}(\overline{z})\ . \tag{6.3.12}$$

Proof. This is Lemma 6.2 of [5] ■

§6.4 The Lebedev transform $\mathbf{L}^\omega_{\ell,q}$ and auxilliary test functions.

Let $0 \neq \omega \in \mathbb{C}$, let $\ell, q \in \mathbb{Z}$ satisfy $\ell \geq |q|$, and let $\rho : G \to \mathbb{C}$ be the function defined in (6.2.1). Following Bruggeman and Motohashi [5], we define

$$P_{\ell,q}(N\backslash G,\omega) = \bigcup_{\varepsilon\in(0,1]}\left\{f \in C^\infty(N\backslash G,\omega) : f \text{ is of } K\text{–type } (\ell,q) \text{ and } \sup_{g\in G}\frac{|f(g)|e^{\varepsilon|\log\rho(g)|}}{\rho(g)} < \infty\right\} \tag{6.4.1}$$

(this being equivalent to Equation (7.3) of [5]). Let $f_\omega \in P_{\ell,q}(N\backslash G,\omega)$. Then $f_\omega \in C^\infty(N\backslash G,\omega)$, and there exist $\sigma_0, R_0, \sigma_\infty, R_\infty \in (0,\infty)$ such that the conditions (6.2.5) and (6.2.22) are satisfied; such a choice of σ_0, R_0, σ_∞ and R_∞ is assumed in what remains of this paragraph. In Equation (7.4) of [5] Bruggeman and Motohashi define their 'Lebedev transform' $\mathbf{L}^\omega_{\ell,q}f_\omega : \{\nu\in\mathbb{C} : |\mathrm{Re}(\nu)| < \sigma_0\}\times\{p\in\mathbb{Z} : |p|\leq\ell\}\to\mathbb{C}$ by

$$\left(\mathbf{L}^\omega_{\ell,q}f_\omega\right)(\nu,p) = \frac{1}{\pi^2\left\|\Phi^\ell_{p,q}\right\|_K}\left\langle f_\omega\ ,\ (\pi|\omega|)^{\overline{\nu}}\left(\frac{-i\omega}{|\omega|}\right)^p\Gamma\left(\ell+1-\overline{\nu}\right)\mathbf{J}_\omega\varphi_{\ell,q}\left(-\overline{\nu},p\right)\right\rangle_{N\backslash G}\ , \tag{6.4.2}$$

where $\|\Phi\|_K = \sqrt{(\Phi,\Phi)_K}$ is the norm associated with the inner product defined in Equation (1.2.22), while the inner product $\langle f, F\rangle_{N\backslash G}$ is that defined in (6.2.9). Given (6.2.5), (6.2.22) and the expansion of $(\mathbf{J}_\omega\varphi_{\ell,q}(\nu,p))(g)$ obtained in Lemma 5.1 of [5], well known estimates for the relevant Bessel functions enable one to show that (6.4.2) defines, for each $p \in \{-\ell, 1-\ell, \ldots, \ell\}$, a function $\nu \mapsto (\mathbf{L}^\omega_{\ell,q}f_\omega)(\nu,p)$ which is holomorphic in the open strip $|\mathrm{Re}(\nu)| < \sigma_0$. By the functional equation (1.7.17), one has the identity $(\mathbf{L}^\omega_{\ell,q}f_\omega)(\nu,p) = (\mathbf{L}^\omega_{\ell,q}f_\omega)(-\nu,-p)$.

We now describe Bruggeman and Motohashi's one-sided inversion, in Theorem 7.1 of [5], of their Lebedev transform operator $\mathbf{L}^\omega_{\ell,q}$. Let $\sigma > 1$, let $S_\sigma = \{\nu\in\mathbb{C} : |\mathrm{Re}(\nu)|\leq\sigma\}$, and define the space $\mathcal{T}^\ell_\sigma$ of 'test functions' to be the linear space of those of the functions

$$\eta : S_\sigma\times\{p\in\mathbb{Z} : |p|\leq\ell\}\to\mathbb{C} \tag{6.4.3}$$

that satisfy all of the following three conditions:

(T1) $\eta(\nu,p) = \eta(-\nu,-p)$;

(T2) for $p \in \{-\ell, 1-\ell, \ldots, \ell\}$, the function $\nu \mapsto \eta(\nu,p)$ can be holomorphically continued into a neighbourhood of the strip S_σ;

(T3) for all $A>0$, one has $\eta(\nu,p) \ll_{\eta,A} (1+|\mathrm{Im}(\nu)|)^{-A} e^{-(\pi/2)|\mathrm{Im}(\nu)|}$.

It is shown by Theorem 7.1 of [5] that, when $\eta \in \mathcal{T}_\sigma^\ell$, one may define $\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta : G \rightarrow \mathbb{C}$ (the 'inverse Lebedev transform') by putting, for $g \in G$,

$$\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta\right)(g) = \frac{1}{2\pi^3 i} \sum_{|p|\leq \ell} \frac{(-i\omega/|\omega|)^p}{\left\|\Phi_{p,q}^{\ell}\right\|_K} \int_{(0)} \eta(\nu,p)(\pi|\omega|)^{-\nu}\Gamma(\ell+1+\nu)\left(\mathbf{J}_\omega \varphi_{\ell,q}(\nu,p)\right)(g)\, \nu^{\epsilon(p)} \sin(\pi\nu) \mathrm{d}\nu \ , \quad (6.4.4)$$

where

$$\epsilon(p) = \begin{cases} 1 & \text{if } p=0; \\ -1 & \text{otherwise.} \end{cases} \quad (6.4.5)$$

One has, in particular, the following theorem.

Theorem 6.4.1 (Bruggeman and Motohashi). *Let $\rho : G \rightarrow (0,\infty)$ be given by (6.2.1), let $0 \neq \omega \in \mathbb{C}$, and let $\mathbf{M}_\omega = \mathbf{M}_\omega^{1,0}$ be the Goodman-Wallach operator on the space $H(1,0)$ (defined as in Subsection 6.3). Suppose moreover that $\ell, q \in \mathbb{Z}$ satisfy $\ell \geq |q|$, and that one has $\sigma > 1$ and $\eta \in \mathcal{T}_\sigma^\ell$. Put*

$$b(\eta) = b(\omega;\ell,q;\eta) = \begin{cases} -2\pi|\omega|\ell(\ell!) \left\|\Phi_{1,q}^{\ell}\right\|_K^{-1} \eta(0,1) & \textit{if } \ell \geq 1; \\ 0 & \textit{otherwise.} \end{cases} \quad (6.4.6)$$

Then

$$\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta \in P_{\ell,q}(N\backslash G,\omega) \ , \quad (6.4.7)$$

$$\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta\right)(g) \ll_{\omega,\eta,A} (\rho(g))^{-A} \qquad (A \in [-2,\infty),\ g \in G) \quad (6.4.8)$$

and, when $g \in G$,

$$\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta\right)(g) = b(\eta)\left(\mathbf{M}_\omega \varphi_{\ell,q}(1,0)\right)(g) + O_{\omega,\eta}\Big((\rho(g))^{\min\{1+\sigma,3\}}\Big) \qquad \textit{if } \rho(g) \leq 1. \quad (6.4.9)$$

Moreover, for all $(\nu,p) \in \mathbb{C} \times \{-\ell, 1-\ell, \ldots, \ell\}$ such that $|\mathrm{Re}(\nu)| < 1$, one has

$$\left(\mathbf{L}_{\ell,q}^{\omega}\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta\right)(\nu,p) = -\frac{2}{\pi}\,\Gamma(\ell+1-\nu)\Gamma(\ell+1+\nu)\,\frac{\sin(\pi\nu)}{\pi\nu}\,\frac{\nu^{1+\epsilon(p)}}{(\nu^2-p^2)}\,\eta(\nu,p) \ , \quad (6.4.10)$$

where $\epsilon(p)$ is given by Equation (6.4.5).

Proof. The results (6.4.7) and (6.4.10) are contained in Theorem 7.1 of [5] (see also the extension of that theorem obtained in Theorem 9.1.4 of [32]).

The result (6.4.9) is a corollary of the combination of Equation (7.14) of [5] and the calculation implicit in [5], (7.15): note that the the conditions (T2) and (T3) on $\eta \in \mathcal{T}_\sigma^\ell$, and the bound (6.3.9) of Lemma 6.3.1, enable one to substitute $\min\{\sigma,2\}$ for the α in Equation (7.14) of [5], and so to deduce that the first sum on the right-hand side of that equation is not greater than $O_{\omega,\eta}\big((\rho(g))^{1+\min\{\sigma,2\}}\big)$ when $\rho(g) \leq 1$. The definition of the constant $b(\eta)$ given in (6.4.6) represents a slight correction of the corresponding definition below [5], (7.15). We have used (6.3.3) and the equations (6.3.7) and (6.3.8) of Lemma 6.3.1 in performing our own check upon the calculation (7.15) of [5].

The case $\rho(g) \leq 1$ of (6.4.8) is a corollary of (6.4.9), (6.4.6) and the case $\nu = 1$, $p = 0$ of the estimate (6.3.7) of Lemma 6.3.1. The case $\rho(g) \geq 1$ of (6.4.8) is a consequence of the equations (5.26)-(5.27) of Lemma 5.1 of [5], and is noted within the proof of Theorem 7.1 of [5]: see the discussion around the bound (7.12) of [5] for the Bessel function $K_\xi(u)$ ■

In [5], Lemma 7.1 and Lemma 7.2, Bruggeman and Motohashi investigate the inverse Lebedev transform $\widetilde{\mathbf{L}}_{\ell,q}^{\omega}$ further. We reproduce those two lemmas here (without the proofs), since the results they contain are

needed for the proof of the spectral Kloosterman sum formula (Theorem B). Before stating these lemmas we clarify that henceforth $L^2(N\backslash G)$ denotes the space of those measurable functions $f : G \to \mathbb{C}$ that satisfy $f(ng) = f(g)$, for all $n \in N$, $g \in G$, and are such that $\int_{N\backslash G} |f(g)|^2 \,\mathrm{d}\dot{g} < \infty$, where the measure $\mathrm{d}\dot{g}$ is that which occurs in (6.2.9); the inner product defined in (6.2.9) makes $L^2(N\backslash G)$ a Hilbert space.

Lemma 6.4.2 (Bruggeman and Motohashi). *Let $\ell \in \mathbb{N} \cup \{0\}$ and $\sigma \in (1, \infty)$, and let $\eta, \theta \in \mathcal{T}_\sigma^\ell$. Then $|\widetilde{\mathbf{L}}^\omega_{\ell,q}\eta| \in L^2(N\backslash G)$ and, for $0 \neq \omega \in \mathbb{C}$ and $q = -\ell, -\ell+1, \ldots, \ell$, one has*

$$\left\langle \widetilde{\mathbf{L}}^\omega_{\ell,q}\eta \,,\, \widetilde{\mathbf{L}}^\omega_{\ell,q}\theta \right\rangle_{N\backslash G} = -\frac{1}{\pi i} \sum_{|p|\leq \ell} \int_{(0)} \eta(\nu,p)\,\overline{\theta(\nu,p)}\;\Gamma(\ell+1-\nu)\Gamma(\ell+1+\nu)\,\frac{\sin^2(\pi\nu)}{\pi^2\nu^2}\,\frac{\nu^{2+2\epsilon(p)}}{(\nu^2-p^2)}\,\mathrm{d}\nu\ , \quad (6.4.11)$$

where $\langle f, F\rangle_{N\backslash G}$ and $\epsilon(p)$ are as defined in (6.2.9) and (6.4.5).

Lemma 6.4.3 (Bruggeman and Motohashi). *Let $\ell \in \mathbb{N} \cup \{0\}$, $\sigma \in (1, 2)$ and $\omega_1, \omega_2, c \in \mathbb{C} - \{0\}$, let $S_\sigma = \{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\}$, and let $\kappa(\omega_1,\omega_2;c)$ be that mapping from $\mathcal{T}_\sigma^\ell$ into the space of all functions $f : S_\sigma \times \{p \in \mathbb{Z} : |p| \leq \ell\} \to \mathbb{C}$ which is given by*

$$\left(\kappa\left(\omega_1,\omega_2;c\right)\eta\right)(\nu,p) = \mathcal{K}_{\nu,p}\left(2\pi\sqrt{\omega_1\omega_2}/c\right)\eta(\nu,p) \qquad (\eta \in \mathcal{T}_\sigma^\ell,\ \nu \in S_\sigma,\ p \in \mathbb{Z} \text{ and } |p| \leq \ell), \qquad (6.4.12)$$

where $\mathcal{K}_{\nu,p}(u) \in \mathbb{C}$ is defined by the equations (1.9.4)-(1.9.6) of Theorem B. Then $\kappa(\omega_1,\omega_2;c)$ is a linear operator from $\mathcal{T}_\sigma^\ell$ into $\mathcal{T}_\sigma^\ell$, and, for $\eta \in \mathcal{T}_\sigma^\ell$, $q = -\ell, 1-\ell, \ldots, \ell$ and $g \in G$, one has (with $\mathbf{h}_u$ as in (1.5.7)):

$$\left(\mathbf{J}_{\omega_2}\mathbf{h}_{1/c}\widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta\right)(g) = \pi^2|c|^{-2}\left(\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q}\kappa(\omega_1,\omega_2;c)\eta\right)(g)\ . \qquad (6.4.13)$$

Remark 6.4.4. The first result of Lemma 6.4.3 (i.e. that $\kappa(\omega_1,\omega_2;c)\eta \in \mathcal{T}_\sigma^\ell$ for all $\eta \in \mathcal{T}_\sigma^\ell$) has been taken from Lemma 9.1.8 of [32]: its proof in respect of cases where one has $1 < \sigma < 3/2$ is indicated there (see, in particular, the upper bound on $|\mathcal{K}_{\nu,p}(z)|$ obtained in Lemma 9.1.7 of [32]), and a trivial extension of this proof yields the required result in the remaining cases, where $3/2 \leq \sigma < 1$. The proof of (6.4.13) is, in part, an application of Lemma 6.3.2 and the identity (6.3.6).

§6.5 Poincaré series revisited.

It is to be assumed throughout this subsection that ω, σ, η, $\mathfrak{a}$, $\mathfrak{b}$, $g_\mathfrak{a}$, $g_\mathfrak{b}$ and the K-type (ℓ, q) are given, with $0 \neq \omega \in \mathfrak{O}$, $\sigma \in (1, 2)$, $\ell, q \in \mathbb{Z}$, $|q| \leq \ell$ and $\eta \in \mathcal{T}_\sigma^\ell$ (the space defined in, and below, (6.4.3)), and with $\mathfrak{a}, \mathfrak{b} \in \mathbb{Q}(i) \cup \{\infty\}$ and $g_\mathfrak{a}, g_\mathfrak{b} \in G$ such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} \in \{\mathfrak{a}, \mathfrak{b}\}$. We suppose also that $\rho : G \to \mathbb{C}$ is the function given by (6.2.1).

Were it guaranteed to be absolutely convergent, for all $g \in G$, the Poincaré series $(P^\mathfrak{a}\widetilde{\mathbf{L}}^\omega_{\ell,q}\eta)(g)$ might, by itself, serve as the principal 'fulcrum' in a proof of the spectral sum formula, Theorem B: for the related identities (6.4.10), (6.4.11) and (6.4.13) are key results for the proof of Theorem B that we are going to describe. However, in view of (6.4.6), the estimate (6.4.9) and the case $\nu = 1$, $p = 0$ of (6.3.7), it follows by [11], Corollary 3.1.6 and Proposition 3.2.1 (2), that the series $(P^\mathfrak{a}\widetilde{\mathbf{L}}^\omega_{\ell,q}\eta)(g)$ will be absolutely convergent, for all $g \in G$, only if one has either $\ell = 0$, or else $\ell \geq 1$ and $\eta(0, 1) = 0$. This inconvenient fact led Bruggeman and Motohashi to construct, in the equations (9.7) and (9.8) of [5], a suitable 'substitute' for $P^\infty\widetilde{\mathbf{L}}^\omega_{\ell,q}\eta$. The construction of this 'substitute' is a critical step in their proof of Theorem 10.1 of [5] (the spectral sum formula for $PSL(2,\mathfrak{O})\backslash PSL(2,\mathbb{C})$). In this subsection we adapt the method of Bruggeman and Motohashi in defining the corresponding 'substitute' $P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^\omega_{\ell,q}\eta$ for the Poincaré series $P^\mathfrak{a}\widetilde{\mathbf{L}}^\omega_{\ell,q}\eta$.

We remark that the choice of Poincaré series in Section 9 of [5] is acknowledged by Bruggeman and Motohashi to have been influenced by the widely applicable method which Miatello and Wallach developed in [33] and [34]. Consequently our work here has been indirectly influenced by those papers of Miatello of Wallach. In particular, we might have obtained our Lemma 6.5.16 by an application of Theorem 2.5 of [33] and the result (1.4.15) of Kim and Shahidi: Lokvenec-Guleska has shown how to do this when $\mathfrak{a} = \infty$ (see

Section 9.2 of [32]). We have instead chosen to rely on bounds for generalised Kloosterman sums (i.e. the results of Subsection 6.1). This is essentially what Bruggeman and Motohashi chose to do in Section 9 of [5]; it enables us to give proofs which are (on the whole) more self-contained than would be the case if we had opted to derive our results from the results of Miatello and Wallach.

We choose, once and for all, a function $\tau \in C^\infty(G)$ such that, for $n \in N$, $r > 0$ and $k \in K$, one has

$$[0,1] \ni \tau(na[r]k) = \tau(a[r]) = \begin{cases} 1 & \text{if } r \leq 1; \\ 0 & \text{if } r \geq 2. \end{cases} \tag{6.5.1}$$

Then, motivated by the estimate (6.4.9), we put

$$\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta = \widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta - b(\omega;\ell,q;\eta)\,\tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0)\,, \tag{6.5.2}$$

where $b(\omega;\ell,q;\eta)$ is the constant in (6.4.6), and where

$$\left(\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g) = \tau(g)\left(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g) \qquad \text{for } \nu \in \mathbb{C},\ g \in G \tag{6.5.3}$$

(we shall use the similar notation $(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ to denote the function $g \mapsto (1-\tau(g))(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$). Note that, by (6.5.2) and (6.4.6), one has $\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta = \widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta$ if $\ell = 0$, or if $\ell \geq 1$ and $\eta(0,1) = 0$. Both the transform $(\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta)(g)$ and the absolute value of this transform may be used in constructing Poincaré series: for, by (6.5.2), the result (6.4.7) of Theorem 6.4.1, our choice of $\tau \in C^\infty(G)$ (satisfying (6.5.1)), the observation preceding (6.3.4) and the definitions (1.4.7) and (1.4.3), it follows that

$$\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta \in C^\infty(N\backslash G,\omega) \qquad \text{and} \qquad \left|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta\right| \in C^0(N\backslash G,0)\,. \tag{6.5.4}$$

Using what is effectively the same construction as occurs in the equations (9.7)-(9.9) of [5], we now define the function $P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta : G \to \mathbb{C}$ by putting, for $g \in G$,

$$\left(P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta\right)(g) = \lim_{\nu \to 1+}\left(P^{\mathfrak{a}}\left(\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)\,\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)\right)(g)\,, \tag{6.5.5}$$

where (as we shall henceforth suppose) the function $\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ is given by (6.5.3). We have, therefore, to show that the limit in Equation (6.5.5) exists. This will be achieved by means of the analytic continuation, with respect to the complex variable ν, of the Poincaré series $(P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$. We also aim to show that the 'pseudo Poincaré series' $P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta$ given by (6.5.5) has the specific properties that will enable us to make use of it in proving Theorem B.

We find some new terminology convenient in stating the lemmas which follow. The space of all of the measurable Γ-automorphic functions $f : G \to \mathbb{C}$ that are essentially bounded is denoted by $L^\infty(\Gamma\backslash G)$. We define also:

$$L^\infty(\Gamma\backslash G;\ell,q) = \{f \in L^\infty(\Gamma\backslash G) : f \text{ is of } K\text{-type } (\ell,q)\}\,. \tag{6.5.6}$$

Since $\mathrm{vol}(\Gamma\backslash G) < \infty$, one has

$$L^{p_1}(\Gamma\backslash G) \supseteq L^{p_2}(\Gamma\backslash G) \qquad (1 \leq p_1 \leq p_2 \leq \infty), \tag{6.5.7}$$

where $L^p(\Gamma\backslash G)$ denotes the space of those of the measurable and Γ-automorphic functions $f : G \to \mathbb{C}$ which are such that $\int_{\Gamma\backslash G} |f|^p \,\mathrm{d}g < \infty$.

Lemma 6.5.1. *Let* $\mathrm{Re}(\nu) > 1$. *Then* $P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) \in C^0(\Gamma\backslash G)$, $P^{\mathfrak{a}}|\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)| \in C^0(\Gamma\backslash G)$ *and* $P^{\mathfrak{a}}|\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)| \in L^\infty(\Gamma\backslash G)$.

Proof. Let $F = \mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$. By the observation preceding (6.3.4), $F \in W_\omega(\Upsilon_{\nu,0};\ell,q) \subset C^0(N\backslash G,\omega)$, and so, given the choice of τ in and above (6.5.1), we have also $|\tau F| \in C^0(N\backslash G,0)$. The estimate (6.3.7) of Lemma 6.3.1 shows, moreover, that the case $\sigma_0 = \mathrm{Re}(\nu)$, $R_0 = 1$ (say) of the condition (6.2.5) is satisfied when $f_\omega = F$; and obviously the same is true if one substitutes $|\tau F|$ for F (since $|\tau(g)| \leq 1$ for all $g \in G$).

Therefore, given that we have $\mathrm{Re}(\nu) > 1$, the hypotheses of Lemma 6.2.2 are satisfied when one takes there: $\sigma_0 = \mathrm{Re}(\nu)$, $R_0 = 1$ and either $f_\omega = F$, or $\omega = 0$ and $f_0 = |\tau F|$. Hence the first two results of the lemma follow by virtue of Corollary 6.2.3.

We now have only to prove that $P^{\mathfrak{a}}|\tau F| \in L^\infty(\Gamma\backslash G)$. Given the results of the lemma that have already been proved, it suffices to show that the function $P^{\mathfrak{a}}|\tau F| : G \to [0,\infty)$ is bounded. One may deduce this from Corollary 6.2.10: for (6.5.1) trivially implies that $(\tau F)(g) = \tau(g)F(g) = 0F(g) = 0$ for all $g \in G$ such that $\rho(g) \geq 2$, and so the condition (6.2.22) is satisfied when $f_\omega = |\tau F|$, $R_\infty = 2$ and $\sigma_\infty = 1$ (say) ∎

Lemma 6.5.2. *For $\nu \in \mathbb{C}$, the function $P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) : G \to \mathbb{C}$ is defined, lies in $C^\infty(\Gamma\backslash G)$, and is of K-type (ℓ,q). The mapping $(\nu,g) \mapsto (P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$ is a continuous function on $\mathbb{C}\times G$; and, for each $g \in G$, the complex function $\nu \mapsto (P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$ is entire. One has, moreover,*

$$\left(P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g_{\mathfrak{b}}g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma \\ \gamma\mathfrak{b} = \mathfrak{a}}} \left(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g\right) \quad \textit{if } \rho(g) \geq 2. \tag{6.5.8}$$

Proof. Let $\nu \in \mathbb{C}$. Put $f_\omega = (1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$, where the meaning of the term '$(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$' is that indicated below (6.5.3). Then, given our choice of τ (described in, and above, (6.5.1)), it follows by the observation preceding (6.3.4) that we have:

$$f_\omega \in C^\infty(N\backslash G,\omega) \subset C^\infty(G)\ , \qquad f_\omega \text{ is of } K\text{-type } (\ell,q) \tag{6.5.9}$$

and

$$f_\omega(g) = \begin{cases} \left(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g) & \text{if } \rho(g) \geq 2; \\ 0 & \text{if } \rho(g) \leq 1. \end{cases} \tag{6.5.10}$$

By (1.5.4), we have also

$$\left(P^{\mathfrak{a}} f_\omega\right)(g_{\mathfrak{b}}g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma} f_\omega\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g\right)\ , \tag{6.5.11}$$

for any $g \in G$ such that the sum on the right-hand side of (6.5.11) converges. Suppose that $Z_1 > 0$ and $R_1 > R_0 > 0$. Put $U = \{(z,r) \in \mathbb{H}_3 : |z| \leq Z_1 \text{ and } R_0 \leq r \leq R_1\}$ and $\tilde{U} = \{n[z]a[r] : (z,r) \in U\}K$. Then, when $g \in \tilde{U}$, it follows by (6.5.10) and Lemma 6.2.1 that each non-zero term occurring in the sum in (6.5.11) corresponds to a γ contained in the set $(\Gamma'_{\mathfrak{a}}\backslash I_0) \cup (\Gamma'_{\mathfrak{a}}\backslash I_1(g))$, where $I_0 = {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0) = \{\gamma \in \Gamma : \gamma\mathfrak{b} = \mathfrak{a}\}$ (so that $\Gamma'_{\mathfrak{a}}\backslash I_0$ is a finite set, empty unless $\mathfrak{a} \stackrel{\Gamma}{\sim} \mathfrak{b}$), while $I_1(g) = \{\gamma \in \Gamma : \gamma\mathfrak{b} \neq \mathfrak{a} \text{ and } 1 < \rho(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g) \leq 1/R_0\}$.

If, in particular, $R_0 = 2$ and $g \in \tilde{U}$, then the equality in (6.5.8) holds: for in this case the set $I_1(g)$ is empty, and it moreover follows by Lemma 6.2.1 and (6.5.10) that for all $\gamma \in I_0 = {}^{\mathfrak{a}}\Gamma^{\mathfrak{b}}(0)$ one has $\rho(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g) = \rho(g) \geq R_0 = 2$, so that the first case of (6.5.10) applies to $f_\omega(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g)$. Since $G = NAK$, and since the assumptions concerning Z_1, R_0 and R_1 are just that $Z_1 > 0$ and $R_1 > R_0 = 2$, the above therefore completes the proof of (6.5.8).

We now revert to considering the more general case of any $Z_1 > 0$, and any R_1 and R_0 with $R_1 > R_0 > 0$. However, since the results that remain to be proved are independent of the cusp $\mathfrak{b}$, it will be convenient to be more specific in another respect, by assuming henceforth that $\mathfrak{b} = \infty$, and that $g_{\mathfrak{b}} = g_\infty = h[1]$.

Returning to the matters discussed in the paragraph containing Equation (6.5.11), we note that each $\gamma \in \Gamma'_{\mathfrak{a}}\backslash I_1(g)$ represents a coset $\Gamma'_{\mathfrak{a}}\gamma_*$ (say), where $\gamma_* \in I_1(g)$ may (by (1.1.20)-(1.1.21) for $\mathfrak{c} = \mathfrak{a}$) be chosen so that one has:

$$g_{\mathfrak{a}}^{-1}\gamma_* g_\infty g \in \tilde{V} = \{n[z]a[r] : (z,r) \in V\}\, K\ ,$$

where $V = \{(z,r) \in \mathbb{H}_3 : |\mathrm{Re}(z)|, |\mathrm{Im}(z)| \leq 1/2 \text{ and } 1 \leq r \leq 1/R_0\}$. Hence, when $g \in \tilde{U}$, the summation in (6.5.11) is effectively over all $\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma$ that are either contained in the finite set $\Gamma'_{\mathfrak{a}}\backslash I_0$, or else represent cosets of the form $\Gamma'_{\mathfrak{a}}\gamma_*$ with $\gamma_* \in \Gamma$ such that $(\gamma_* g_\infty \tilde{U}) \cap (g_{\mathfrak{a}}\tilde{V}) \neq \emptyset$. Given the definition of the action of G upon $\mathbb{H}_3$, and given that $g_\infty = h[1]$ (the identity element of G), the latter condition is equivalent to the condition that $(\gamma_* U) \cap (g_{\mathfrak{a}} V) \neq \emptyset$, and so it is satisfied only if one has

$$(\gamma_* W) \cap W \neq \emptyset\ , \tag{6.5.12}$$

where $W = U \cup (g_{\mathfrak{a}} V)$. Since this set W is a compact subset of $\mathbb{H}_3$, and since (by Theorem 2.1.2 of [11]) the discrete group $\Gamma < SL(2,\mathbb{C})$ is discontinuous, it follows that the condition (6.5.12) is satisfied for at most finitely many choices of $\gamma_* \in \Gamma$. The condition (6.5.12) is, furthermore, independent of the variable $g \in \tilde{U}$ (as is the finite set $\Gamma'_{\mathfrak{a}} \backslash I_0$). Therefore (and since we have $g_\infty g = h[1]g = g$ for all $g \in G$) there exists a finite set $\{\gamma_1, \ldots, \gamma_J\} \subset \Gamma$ such that

$$(P^{\mathfrak{a}} f_\omega)(g) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{j=1}^{J} f_\omega\left(g_{\mathfrak{a}}^{-1} \gamma_j g\right) \qquad \text{for all } g \in \tilde{U}. \tag{6.5.13}$$

Since $G = NAK$, and since $Z_1 > 0$ and $R_1 > R_0 > 0$ are arbitrary, the fact that we obtain (6.5.13) is enough to prove that, for each $g \in G$, the Poincaré series $(P^{\mathfrak{a}} f_\omega)(g)$ is absolutely convergent (by virtue of it being a series that contains only finitely many non-zero terms). The mapping $g \mapsto (P^{\mathfrak{a}} f_\omega)(g)$ is therefore (given the definition (1.5.4) and the first part of (6.5.9)) a well-defined Γ-automorphic function on G. Similarly, since it follows by (6.5.13), (6.5.9), (1.3.1) and the left-invariance of all elements of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k})$ that the restriction of $P^{\mathfrak{a}} f_\omega$ to the open set $\ddot{U} = \mathrm{Int}(\tilde{U})$ is a smooth function (i.e. an element of $C^\infty(\ddot{U})$) satisfying $(\mathbf{H}_2 P^{\mathfrak{a}} f_\omega)(g) = -iq(P^{\mathfrak{a}} f_\omega)(g)$ and $(\Omega_{\mathfrak{k}} P^{\mathfrak{a}} f_\omega)(g) = -\frac{1}{2}\left(\ell^2 + \ell\right)(P^{\mathfrak{a}} f_\omega)(g)$ for $g \in \ddot{U}$, we may deduce that $P^{\mathfrak{a}} f_\omega$ is in fact a smooth function on G of K-type (ℓ, q).

By (6.5.13) one moreover obtains, as a corollary of the expansion of $(\mathbf{M}_\omega \varphi_{\ell,q}(\nu,p))(g)$ in terms of Bessel functions that is given by the equations (6.13) and (6.14) of Lemma 6.1 of [5], the result that, for each $g \in G$, the mapping $\mu \mapsto (P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega \varphi_{\ell,q}(\mu,0))(g)$ is an entire complex function. The same expansion of $(\mathbf{M}_\omega \varphi_{\ell,q}(\nu,p))(g)$ in terms of Bessel function also enables one to show, in particular, that the mapping $(\mu, g) \mapsto (\mathbf{M}_\omega \varphi_{\ell,q}(\mu,0))(g)$ is a continuous function on $\mathbb{C} \times G$. Hence and by (6.5.13) one finds that, since G is a topological group, and since $(1-\tau) \in C^\infty(G)$, the mapping $(\mu, g) \mapsto (P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega \varphi_{\ell,q}(\mu,0))(g)$ is a continuous function on $\mathbb{C} \times G$ ■

Lemma 6.5.3. *Let $0 \neq \omega' \in \mathbb{C}$, and let $p \in \mathbb{Z}$ satisfy $|p| \leq \ell$. Then, for all $g \in G$, each of the two mappings $\nu \mapsto (\mathbf{J}_{\omega'} \varphi_{\ell,q}(\nu,p))(g)$ and $\nu \mapsto (\mathbf{M}_{\omega'} \varphi_{\ell,q}(\nu,p))(g)$ is an entire function of the complex variable ν. Suppose moreover that $r_0 \in (0,\infty)$, $\nu \in \mathbb{C}$, $|\mathrm{Re}(\nu)| \leq \sigma_1 < \infty$ and $\omega' \in \mathfrak{O}$ (so that, in particular, $|\omega'| \geq 1$), and that $n \in N$, $r > 0$, $k \in K$ and $g = na[r]k$ (so that $\rho(g) = r$). Put*

$$f_{\omega'}(\nu, p; g) = (\pi|\omega'|)^{-\nu} (i\omega'/|\omega'|)^p \Gamma(\ell + 1 + \nu)\big(\mathbf{J}_{\omega'} \varphi_{\ell,q}(\nu,p)\big)(g) \;.$$

Then

$$f_{\omega'}(\nu, p; g) \ll_{\ell,\sigma_1,r_0} |\omega'|^{-1} (1 + |\mathrm{Im}(\nu)|)^{\ell - |p|} |\omega' r|^{\ell+1} e^{-2\pi|\omega'|r} \qquad \text{if } |\omega'|r \geq r_0. \tag{6.5.14}$$

Moreover, for all $\varepsilon \in (0, 1/4]$ and all $d \in \mathbb{N}$ such that $d/2 > \sigma_1 + \ell$, one has

$$f_{\omega'}(\nu,p;g) = \frac{r}{e^{(\pi/2)|\mathrm{Im}(\nu)|}} \cdot \begin{cases} O_{\ell,\sigma_1,r_0,\varepsilon}\Big((1 + |\mathrm{Im}(\nu)|)^{|\mathrm{Re}(\nu)| - 1/2 + \ell} |\omega' r|^{-|\mathrm{Re}(\nu)| - \varepsilon}\Big) & \text{if } |\omega'|r \leq r_0, \\ O_{d,\sigma_1,r_0}\Big((1 + |\mathrm{Im}(\nu)|)^{\mathrm{Re}(\nu) - 1/2 + \ell + d} |\omega' r|^{-\mathrm{Re}(\nu) + \ell + |p| - d}\Big) & \text{if } |\omega'|r \geq r_0. \end{cases} \tag{6.5.15}$$

The case $|\omega'|r \geq r_0$ of (6.5.15) remains valid if, in place of the O-term appearing there, one substitutes the term $O_{d,\sigma_1,r_0}(\min\{(1 + |\mathrm{Im}(\nu)|)^{-\mathrm{Re}(\nu) - 1/2 + \ell + d} |\omega' r|^{\mathrm{Re}(\nu) + \ell + |p| - d}, (1 + |\mathrm{Im}(\nu)|)^{-1/2 + \ell + d} |\omega' r|^{\ell + |p| - d}\})$.

Proof. The first assertion, concerning the mappings $\nu \mapsto (\mathbf{J}_{\omega'} \varphi_{\ell,q}(\nu,p))(g)$, $\nu \mapsto (\mathbf{M}_{\omega'} \varphi_{\ell,q}(\nu,p))(g)$, is a corollary of the relevant expansions in terms of Bessel functions $K_\mu(2\pi r)$ and $I_\mu(2\pi r)$ obtained in [5], Lemma 5.1 and Lemma 6.1.

The result (6.5.15) is proved similarly to the equation (4.28) of Lemma 4.1.3 of [32], and (see our Remark 6.5.4, following this proof) coincides with that result in the respect of cases with $|\omega'|r \leq r_0$ and $\mathrm{Re}(\nu) \leq 0$. Its proof (which we omit) involves the application of the equations (5.26)-(5.27) of Lemma 5.1 of [5], and the estimates

$$K_\mu(2\pi R) \ll_{\varepsilon,\sigma_2,r_0} e^{-(\pi/2)|\mathrm{Im}(\mu)|} (1 + |\mathrm{Im}(\mu)|)^{|\mathrm{Re}(\mu)| - 1/2} R^{-|\mathrm{Re}(\mu)| - \varepsilon} \qquad (R \in (0, r_0],\ |\mathrm{Re}(\mu)| \leq \sigma_2 < \infty)$$

and

$$K_\mu(2\pi R) \ll_{d,\sigma_2} e^{-(\pi/2)|\mathrm{Im}(\mu)|}(1+|\mathrm{Im}(\mu)|)^{\mathrm{Re}(\mu)+d}R^{-\mathrm{Re}(\mu)-d} \qquad (R\in(0,\infty),\ |\mathrm{Re}(\mu)|\le\sigma_2<d/2),$$

where $K_\mu(z)$ is the modified Bessel function defined in the equations 10.27.4-10.27.5 of [38]; the first of these estimates is [32], (1.33); the second is [32], (1.37) (and is also equivalent to [5], (7.12)).

The result stated immediately below (6.5.15) is a consequence of (6.5.15) and the functional equation $f_{\omega'}(\nu,p;g)=f_{\omega'}(-\nu,-p;g)$, which is (1.7.17).

For the proof of (6.5.14), we use (in place of the above estimates for $K_\mu(2\pi R)$) the integral representation

$$K_\mu(2\pi R)=\frac{1}{2}\int_0^\infty e^{-\pi R(t+t^{-1})}t^{-\mu-1}\mathrm{d}t \qquad (R>0),$$

which is derived from Equation 10.32.10 of [38]. This integral representation of $K_\mu(2\pi R)$ implies that

$$e^{2\pi R}\,|K_\mu(2\pi R)|\le\int_1^\infty e^{-\pi Rt(1-t^{-1})^2}t^{M-1}\mathrm{d}t \qquad (R>0 \text{ and } M\ge|\mathrm{Re}(\mu)|+1). \tag{6.5.16}$$

By setting $\tau=\cosh^{-1}(1+1/2\pi R)\in(0,\infty)$, and then applying the inequalities

$$\left(1-t^{-1}\right)^2\ge\begin{cases}1/(\pi Re^\tau) & \text{if } t\ge e^\tau,\\ 0 & \text{if } 1\le t<e^\tau,\end{cases}$$

we deduce from (6.5.16) that

$$|K_\mu(2\pi R)|\le 2^M\left(M^{-1}+\Gamma(M)\right)\exp\left(\frac{M}{2\pi R}-2\pi R\right) \qquad \text{for } |\mathrm{Re}(\mu)|\le M-1 \text{ and } R>0.$$

By combining this last result with the case $\omega=1$ of the equations (5.26)-(5.27) of Lemma 5.1 of [5] we obtain, when $n_1\in N$, $k_1\in K$, $|\mathrm{Re}(\nu)|\le\sigma_1$ and $R\ge r_0$, the upper bound:

$$\pi^{-\nu}i^p\,\Gamma(\ell+1+\nu)\big(\mathbf{J}_1\varphi_{\ell,q}(\nu,p)\big)(n_1a[R]k_1)\ll_{\ell,\sigma_1,r_0}(1+|\mathrm{Im}(\nu)|)^{\ell-|p|}R^{\ell+1}e^{-2\pi R}\,. \tag{6.5.17}$$

It is a property of the Jacquet operator that $|u|^4\mathbf{J}_{u^2\xi}=\mathbf{h}_u\mathbf{J}_\xi\mathbf{h}_u$ for $u\in\mathbb{C}^*$, and so, by Equation (1.8.2) and the linearity of $\mathbf{J}_\xi$, one finds that

$$\big(\mathbf{J}_{\omega'}\varphi_{\ell,q}(\nu,p)\big)(g)=|\omega'|^{\nu-1}\left(\omega'/\,|\omega'|\right)^{-p}\big(\mathbf{J}_1\varphi_{\ell,q}(\nu,0)\big)\big(h\big[\sqrt{\omega'}\,\big]g\big) \qquad \text{for } \nu\in\mathbb{C},\ g\in G. \tag{6.5.18}$$

Since, moreover, $\mathbf{h}_u\rho=|u|^2\rho$ for $u\in\mathbb{C}^*$, the combination of (6.5.17) and (6.5.18) yields (6.5.14) ■

Remark 6.5.4. It follows by Stirling's asymptotic formula for $\log\Gamma(z)$ that one has

$$|\Gamma(\mu+1)|\gg_{\sigma_2}(1+|\mathrm{Im}(\mu)|)^{\mathrm{Re}(\mu)+1/2}e^{-(\pi/2)|\mathrm{Im}(\mu)|} \qquad (\mu\in\mathbb{C},\ |\mathrm{Re}(\mu)|\le\sigma_2<\infty). \tag{6.5.19}$$

Before stating the next lemma, we find it convenient to first define one relevant piece of new terminology. For all $\lambda,\kappa\in\mathbb{Z}$ with $\lambda\ge|\kappa|$, and for all $\theta\in\mathcal{T}_\sigma^\lambda$, we put

$$\widetilde{\mathbf{L}}^{\omega,\dagger}_{\lambda,\kappa}\theta=\widetilde{\mathbf{L}}^{\omega}_{\lambda,\kappa}\theta-b(\omega;\lambda,\kappa;\theta)\mathbf{M}_\omega\varphi_{\lambda,\kappa}(1,0)\,, \tag{6.5.20}$$

where the constant $b(\omega;\lambda,\kappa;\theta)$ is defined as in (6.4.6). It is then an immediate corollary of the relation (6.4.7) of Theorem 6.4.1 and the observation preceding (6.3.4) that, subject to the same hypotheses under which $\widetilde{\mathbf{L}}^{\omega,\dagger}_{\lambda,\kappa}\theta$ was just defined, one has:

$$\widetilde{\mathbf{L}}^{\omega,\dagger}_{\lambda,\kappa}\theta\in C^\infty(N\backslash G,\omega) \quad \text{and} \quad \widetilde{\mathbf{L}}^{\omega,\dagger}_{\lambda,\kappa}\theta \text{ is of } K\text{-type } (\lambda,\kappa). \tag{6.5.21}$$

For later reference we note here also that, given our choice of $\tau \in C^\infty(G)$ (satisfying the conditions in (6.5.1)), it follows by the estimate (6.4.9) of Theorem 6.4.1, and the definitions (6.5.2) and (6.5.20), that if $\lambda, \kappa \in \mathbb{Z}$, $\lambda \ge |\kappa|$ and $\theta \in \mathcal{T}_\sigma^\lambda$ then

$$\big(\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,*}\theta\big)(g) = \big(\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta\big)(g) \ll_{\omega,\theta} (\rho(g))^{1+\sigma} \qquad \text{for all } g \in G \text{ such that } \rho(g) \le 1. \tag{6.5.22}$$

Lemma 6.5.5. *Put*

$$I(\ell, q) = \{(\lambda, \kappa) \in \mathbb{Z} \times \mathbb{Z} \,:\, \lambda \ge |\kappa| \text{ and } \max\{|\lambda - \ell|, |\kappa - q|\} \le 1\} \ . \tag{6.5.23}$$

Then, for each $\mathbf{X} \in \mathfrak{g}$, *there exists a family* $\big([\mathbf{X}]_{\ell,q}^{\lambda,\kappa}\big)_{(\lambda,\kappa)\in I(\ell,q)}$ *of linear operators on the space* $\mathcal{T}_\sigma^\ell$ *such that one has both*

$$[\mathbf{X}]_{\ell,q}^{\lambda,\kappa} : \mathcal{T}_\sigma^\ell \to \mathcal{T}_\sigma^\lambda \ , \qquad \textit{for each } (\lambda, \kappa) \in I(\ell, q), \tag{6.5.24}$$

and

$$\sum_{(\lambda,\kappa)\in I(\ell,q)} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger} [\mathbf{X}]_{\ell,q}^{\lambda,\kappa} \theta = \mathbf{X}\widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger}\theta \ , \qquad \textit{for all } \theta \in \mathcal{T}_\sigma^\ell. \tag{6.5.25}$$

Proof. As in (3.11) of [5], we put

$$\mathbf{H}_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \ , \qquad \mathbf{V}_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \ , \qquad \mathbf{V}_2 = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix} \tag{6.5.26}$$

(having already defined $\mathbf{H}_2$, $\mathbf{W}_1$ and $\mathbf{W}_2$ in (1.2.9)). The set $\mathcal{B} = \{\mathbf{H}_1, \mathbf{H}_2, \mathbf{V}_1, \mathbf{V}_2, \mathbf{W}_1, \mathbf{W}_2\}$ is a basis of the real Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of G, and so is also a $\mathbb{C}$-basis of the complex Lie algebra $\mathfrak{g}$. Another $\mathbb{C}$-basis for $\mathfrak{g}$ is the set $\mathcal{B}_1 = \{\mathbf{H}_1, \mathbf{H}_2, \mathbf{F}^+, \mathbf{F}^-, \mathbf{E}^+, \mathbf{E}^-\}$, where

$$\mathbf{F}^\pm = \mathbf{V}_1 \pm i\mathbf{V}_2 \qquad \text{and} \qquad \mathbf{E}^\pm = \mathbf{W}_1 \pm i\mathbf{W}_2 \tag{6.5.27}$$

(with the factor 'i' here signifying complexification). Therefore we may confine ourselves, in this proof, to a discussion of the cases in which one has $\mathbf{X} \in \mathcal{B}_1$: for, by linearity, these special cases of the lemma imply the general case.

Let $\mathbf{X} \in \mathcal{B}_1$. Then, for $(\nu, p) \in \mathbb{C} \times \mathbb{Z}$ with $|p| \le \ell$, one has

$$\mathbf{X}\mathbf{J}_\omega \varphi_{\ell,q}(\nu, p) = \mathbf{J}_\omega \mathbf{X} \varphi_{\ell,q}(\nu, p) \ , \tag{6.5.28}$$

where, by virtue of the $\mathfrak{g}$-invariance of the space $H(\nu, p)$ defined in Equation (1.6.1), there exist certain complex constants $c_{\ell,q}^{\mathbf{X}}(\lambda, \kappa; \nu, p)$ $(\lambda, \kappa \in \mathbb{Z})$, all but finitely many of which are equal to zero, such that

$$\mathbf{X}\varphi_{\ell,q}(\nu, p) = \sum_{\lambda=|p|}^{\infty} \sum_{\kappa=-\lambda}^{\lambda} c_{\ell,q}^{\mathbf{X}}(\lambda, \kappa; \nu, p) \varphi_{\lambda,\kappa}(\nu, p) \ . \tag{6.5.29}$$

In his preliminary notes preparatory to work reported on in [5] Bruggeman has computed, for each $\mathbf{X} \in \mathcal{B}_1$, the coefficients in (6.5.29). The results of his computations (kindly made available to us by personal communication) are crucial for this proof; they show, in particular, that $c_{\ell,q}^{\mathbf{X}}(\lambda, \kappa; \nu, p)$ is a polynomial function of the complex variable ν, and is identically zero when $(\lambda, \kappa) \notin I(\ell, q)$.

We confine ourselves, in what follows, to a discussion of the single case in which $\mathbf{X} = \mathbf{F}^+$. This is justifiable, since one can deal similarly with the cases in which $\mathbf{X} \in \{\mathbf{H}_1, \mathbf{H}_2, \mathbf{F}^-, \mathbf{E}^+, \mathbf{E}^-\}$. Bruggeman found that

$$c_{\ell,q}^{\mathbf{F}^+}(\lambda, \kappa; \nu, p) = 0 \qquad \text{unless } (\lambda, \kappa) \in I(\ell, q) \text{ and } \kappa = q + 1, \tag{6.5.30}$$

and that, when $(\lambda, q+1) \in I(\ell, q)$ and $(\nu, p) \in \mathbb{C} \times \mathbb{Z}$ is such that $|p| \leq \ell$, one has:

$$c_{\ell,q}^{\mathbf{F}^+}(\lambda, q+1; \nu, p) = \begin{cases} (\ell+1)^{-1}(2\ell+1)^{-1}(\nu+\ell+1)\left((\ell+1)^2 - p^2\right) & \text{if } \lambda = \ell+1; \\ -\ell^{-1}(\ell+1)^{-1}(\ell-q)\nu p & \text{if } \lambda = \ell; \\ \ell^{-1}(2\ell+1)^{-1}(\ell-q)(\ell-q-1)(\ell-\nu) & \text{if } \lambda = \ell-1 \geq |p|; \\ 0 & \text{otherwise.} \end{cases} \tag{6.5.31}$$

Note that in (6.5.31) the case $\lambda = \ell$ will arise only if $\ell \geq |q+1|$, and that this inequality implies $\ell \neq 0$ (since it is assumed that we have $\ell \geq |q|$).

Suppose now that $\theta \in \mathcal{T}_\sigma^\ell$. Motivated by the definition of the transform $\widetilde{\mathbf{L}}_{\ell,q}^\omega$ in Equation (6.4.4) we observe now that it follows by (6.5.28) and (6.5.29), for $\mathbf{X} = \mathbf{F}^+$, and by (6.5.30), (6.5.31), (6.5.23), the substitution $\nu = it$ and the linearity of the Jacquet operator $\mathbf{J}_\omega$, that if $g \in G$ and p is an integer satisfying $|p| \leq \ell$ then one has:

$$\int_{(0)} \theta(\nu, p)(\pi|\omega|)^{-\nu}\Gamma(\ell+1+\nu)\left(\mathbf{F}^+\mathbf{J}_\omega\varphi_{\ell,q}(\nu,p)\right)(g)\,\nu^{\epsilon(p)}\sin(\pi\nu)\mathrm{d}\nu =$$

$$= -\int_{-\infty}^{\infty} \theta(it,p)(\pi|\omega|)^{-it}\,\Gamma(\ell+1+it)\,\times \tag{6.5.32}$$

$$\times\left(\sum_{\lambda=\max\{\ell-1,|p|,|q+1|\}}^{\ell+1} c_{\ell,q}^{\mathbf{F}^+}(\lambda, q+1; it, p)\left(\mathbf{J}_\omega\varphi_{\lambda,q+1}(it,p)\right)(g)\right)(it)^{\epsilon(p)}\sinh(\pi t)\mathrm{d}t\ .$$

Given the condition (T2) below (6.4.3), the expansion of $(\mathbf{J}_\omega\varphi_{\lambda,\kappa}(\nu,p))(g)$ obtained in the equations (5.26)-(5.27) of Lemma 5.1 of [5], the definition (6.4.5) and the equation (6.5.31), one may check that (once any inessential discontinuities at $t=0$ or $t=i$ are removed) the last integrand above is a continuous function $(t,g) \mapsto f(t,g)$ from the set $\{t \in \mathbb{C} : |\mathrm{Im}(t)| \leq \sigma\} \times G$ into $\mathbb{C}$. This integrand $f(t,g)$ is, in particular, continuous on $\mathbb{R} \times G$; by the case $\omega' = \omega,\ \sigma_1 = 1/2,\ \varepsilon = 1/4,\ d \to \infty$ of the bound (6.5.15) of Lemma 6.5.3, the equation (6.5.31) and the condition (T3) below (6.4.3), it moreover satisfies

$$f(t,g) \ll_{\theta,\ell,|\omega|,R,A} (1+|t|)^{\ell+(1/2)+\epsilon(p)-A} \qquad \text{for } A, R \in [1,\infty),\ t \in \mathbb{R} \text{ and } g \in G \text{ such that } \rho(g) \leq R.$$

Since we have here $\epsilon(p) \leq 1$ (by (6.4.5)), it may be deduced from the special case $A = \ell+3$ of the above bound for the integrand $f(t,g)$ that integral on the right-hand side of Equation (6.5.32) converges uniformly for all g lying in any given compact subset of G.

Similarly to the above, it may be shown that that the corresponding 'undifferentiated' integral

$$-\int_{-\infty}^{\infty} \theta(it,p)(\pi|\omega|)^{-it}\,\Gamma(\ell+1+it)\,(\mathbf{J}_\omega\varphi_{\ell,q}(it,p))(g)(it)^{\epsilon(p)}\sinh(\pi t)\mathrm{d}t$$

is absolutely convergent (for $|p| \leq \ell$), and that its integrand $F(t,g)$ (say) is continuous on $\mathbb{R} \times G$.

Given the observations of the preceding two paragraphs, and bearing in mind the definition of the differential operator $\mathbf{F}^+$ (via the cases $\mathbf{X} = \mathbf{V}_1$ and $\mathbf{X} = \mathbf{V}_2$ of (1.2.6), and the first part of (6.5.27)), it follows by the second proposition of Section 1.88 of [43] that one may, by 'differentiating inside the integral', deduce from the equation (6.4.4) (with θ substituted for η) that one has

$$\left(\mathbf{F}^+\widetilde{\mathbf{L}}_{\ell,q}^\omega\theta\right)(g) = \frac{1}{2\pi^3 i}\sum_{|p|\leq\ell}\frac{(-i\omega/|\omega|)^p}{\left\|\Phi_{p,q}^\ell\right\|_K}\int_{(0)}\theta(\nu,p)(\pi|\omega|)^{-\nu}\,\Gamma(\ell+1+\nu)\left(\mathbf{F}^+\mathbf{J}_\omega\varphi_{\ell,q}(\nu,p)\right)(g)\,\nu^{\epsilon(p)}\sin(\pi\nu)\mathrm{d}\nu =$$

$$= \frac{1}{2\pi^3 i}\sum_{\lambda=\max\{\ell-1,|q+1|\}}^{\ell+1}\ \sum_{|p|\leq\min\{\ell,\lambda\}}\frac{(-i\omega/|\omega|)^p}{\left\|\Phi_{p,q}^\ell\right\|_K}\times$$

$$\times\int_{(0)}\theta(\nu,p)(\pi|\omega|)^{-\nu}\,\Gamma(\ell+1+\nu)c_{\ell,q}^{\mathbf{F}^+}(\lambda,q+1;\nu,p)\,(\mathbf{J}_\omega\varphi_{\lambda,q+1}(\nu,p))(g)\,\nu^{\epsilon(p)}\sin(\pi\nu)\mathrm{d}\nu =$$

$$= \sum_{(\lambda,\kappa)\in I(\ell,q)}\left(\widetilde{\mathbf{L}}_{\lambda,\kappa}^\omega\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta\right)(g)\ , \tag{6.5.33}$$

where, for $(\lambda,\kappa)\in I(\ell,q)$, the function $\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta : \{\nu\in\mathbb{C} : |\mathrm{Re}(\nu)|\leq\sigma\}\times\{p\in\mathbb{Z} : |p|\leq\lambda\}\rightarrow\mathbb{C}$ satisfies

$$\left(\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta\right)(\nu,p)=\begin{cases}\dfrac{\|\Phi_{p,\kappa}^{\lambda}\|_K}{\|\Phi_{p,q}^{\ell}\|_K}\,\dfrac{\Gamma(\ell+1+\nu)}{\Gamma(\lambda+1+\nu)}\,c_{\ell,q}^{\mathbf{F}^+}(\lambda,\kappa;\nu,p)\theta(\nu,p) & \text{if } \kappa=q+1 \text{ and } |p|\leq\ell,\\ 0 & \text{otherwise,}\end{cases}\tag{6.5.34}$$

while the summand in (6.5.33) is given by the case $\eta=\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta$, $(\ell,q)=(\lambda,\kappa)$ of the equation (6.4.4).

With regard to the definition (6.5.34) of the function $\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega}\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta$, we observe that if $(\lambda,\kappa)\in I(\ell,q)$ then the function $p\mapsto\|\Phi_{p,\kappa}^{\lambda}\|_K/\|\Phi_{p,q}^{\ell}\|_K$ is (by (1.6.6)) an even function from $\{-\lambda,1-\lambda,\ldots,\lambda\}$ into $(0,\infty)$. Moreover, when $(\lambda,q+1)\in I(\ell,q)$ and $(\nu,p)\in\mathbb{C}\times\mathbb{Z}$ is such that $|p|\leq\min\{\lambda,\ell\}$, it follows by Equation (6.5.31) that

$$\frac{\Gamma(\ell+1+\nu)}{\Gamma(\lambda+1+\nu)}\,c_{\ell,q}^{\mathbf{F}^+}(\lambda,q+1;\nu,p)=\begin{cases}(\ell+1)^{-1}(2\ell+1)^{-1}\left((\ell+1)^2-p^2\right) & \text{if } \lambda=\ell+1;\\ -\ell^{-1}(\ell+1)^{-1}(\ell-q)\nu p & \text{if } \lambda=\ell;\\ \ell^{-1}(2\ell+1)^{-1}(\ell-q)(\ell-q-1)\left(\ell^2-\nu^2\right) & \text{if } \lambda=\ell-1.\end{cases}$$

Hence (and since $|p|$ is an even function of p) one may check that, when $(\lambda,\kappa)\in I(\ell,q)$, the conditions (T1)-(T3) below (6.4.3) continue to hold if one substitutes there $\left(\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta\right)(\nu,p)$ for $\eta(\nu,p)$. Therefore we obtain the case $\mathbf{X}=\mathbf{F}^+$ of the result (6.5.24) stated in the lemma.

Given the result obtained in (6.5.33), and given the definition of $\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta$ in (6.5.20), the case $\mathbf{X}=\mathbf{F}^+$ of the result (6.5.25) will follow if we are able to show that

$$\sum_{(\lambda,\kappa)\in I(\ell,q)} b\Big(\omega;\lambda,\kappa;\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta\Big)\,\mathbf{M}_\omega\varphi_{\lambda,\kappa}(1,0)=b(\omega;\ell,q;\theta)\mathbf{F}^+\mathbf{M}_\omega\varphi_{\ell,q}(1,0)\ .\tag{6.5.35}$$

Since the Goodman-Wallach operator $\mathbf{M}_\omega$ commutes with all elements of the Lie algebra $\mathfrak{g}$, and since we have the case $\mathbf{X}=\mathbf{F}^+$ of the equation (6.5.29), in which the coefficients $c_{\ell,q}^{\mathbf{F}^+}(\lambda,\kappa;\nu,p)$ $(\lambda\geq|p|,\ |\kappa|\leq\lambda)$ are given by (6.5.30)-(6.5.31), it follows by virtue of the linearity of the operator $\mathbf{M}_\omega$ (in combination with the observation preceding (6.3.4), and the definition of 'K-type' in the equations (1.3.1)) that the identity (6.5.35) is valid if and only if

$$b\Big(\omega;\lambda,\kappa;\left[\mathbf{F}^+\right]_{\ell,q}^{\lambda,\kappa}\theta\Big)=b(\omega;\ell,q;\theta)c_{\ell,q}^{\mathbf{F}^+}(\lambda,\kappa;1,0)\tag{6.5.36}$$

for all $(\lambda,\kappa)\in I(\ell,q)$.

Let $(\lambda,\kappa)\in I(\ell,q)$. By the definition (6.4.6) of $b(\omega;\ell,q;\theta)$, and by (6.5.34) and (6.5.30), one finds that both sides of the equality sign in (6.5.36) equal zero if either $\ell=0$ or $\kappa\neq q+1$. If, on the other hand, we have $\ell\geq 1$ and $\kappa=q+1$ (so that $(\lambda,q+1)=(\lambda,\kappa)\in I(\ell,q)$) then it follows from (6.4.6) and (6.5.34) that the equation (6.5.36) holds if and only if it is the case that

$$\lambda c_{\ell,q}^{\mathbf{F}^+}(\lambda,q+1;0,1)=\ell c_{\ell,q}^{\mathbf{F}^+}(\lambda,q+1;1,0)\tag{6.5.37}$$

(note that the factor λ on the left-hand side of this equation makes it unnecessary to distinguish the case $\lambda=0$). Since the equation (6.5.31) allows one to show that if $\ell\geq 1$ then the condition (6.5.37) is satisfied when $(\lambda,q+1)\in I(q,\ell)$ (with only the case $\lambda=0$ requiring any thought at all), we are therefore able to conclude that the condition (6.5.36) is satisfied for all $(\lambda,\kappa)\in I(\ell,q)$; it follows that the identity (6.5.35) is valid, and so our proof of the case $\mathbf{X}=\mathbf{F}^+$ of the lemma is complete. Similar proofs exist for the cases in which $\mathbf{X}\in\{\mathbf{H}_1,\mathbf{H}_2,\mathbf{F}^-,\mathbf{E}^+,\mathbf{E}^-\}=\mathcal{B}_1-\{\mathbf{F}^+\}$. Therefore, and since $\mathcal{B}_1$ is a $\mathbb{C}$-basis of $\mathfrak{g}$, the results stated in the lemma are valid in general (i.e. for all $\mathbf{X}\in\mathfrak{g}$) ■

Remark 6.5.6. The above proof is not as we would ideally like it: for the validation of the identity (6.5.35) is reliant on detailed (and quite mindless) calculations. Despite giving it some thought, we were unable to discover a general principle that might 'explain' the identity (6.5.35).

Lemma 6.5.7. *Let $\ell_1, q_1 \in \mathbb{Z}$ be such that $\ell_1 \geq |q_1|$, let $I(q_1, \ell_1)$ be defined as in Equation (6.5.23) of Lemma 6.5.5, let $\theta \in \mathcal{T}_\sigma^{\ell_1}$, and let $\mathbf{X} \in \mathfrak{g}$. Then $\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta \in C^\infty(N\backslash G, \omega)$. Moreover, the Poincaré series $(P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta)(g)$ and $(P^{\mathfrak{a}}\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta)(g)$ are well-defined and absolutely convergent for all $g \in G$, and one has:*

$$\mathbf{X}P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta = P^{\mathfrak{a}}\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta \tag{6.5.38}$$

and, for some $\theta^{\mathbf{X}} \in \prod_{(\lambda,\kappa)\in I(\ell_1,q_1)} \mathcal{T}_\sigma^{\lambda}$,

$$\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta = \sum_{(\lambda,\kappa)\in I(\ell_1,q_1)} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta^{\mathbf{X}}_{(\lambda,\kappa)} \qquad \text{and} \qquad P^{\mathfrak{a}}\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta = \sum_{(\lambda,\kappa)\in I(\ell_1,q_1)} P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta^{\mathbf{X}}_{(\lambda,\kappa)}\ . \tag{6.5.39}$$

Proof. The result that $\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta \in C^\infty(N\backslash G, \omega)$ is an immediate corollary of the first part of (6.5.21) and the left-invariance of the differential operator $\mathbf{X}$. By Lemma 6.5.5, we moreover have the identity

$$\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta = \sum_{(\lambda,\kappa)\in I(\ell_1,q_1)} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta^{\mathbf{X}}_{(\lambda,\kappa)}\ , \tag{6.5.40}$$

where $\theta^{\mathbf{X}}_{(\lambda,\kappa)} = [X]_{\ell_1,q_1}^{\lambda,\kappa}\theta$ (with the operator $[X]_{\ell_1,q_1}^{\lambda,\kappa}$ as described in that lemma) so that, for $(\lambda,\kappa) \in I(\ell_1, q_1)$,

$$\theta^{\mathbf{X}}_{(\lambda,\kappa)} = [X]_{\ell_1,q_1}^{\lambda,\kappa}\theta \in \mathcal{T}_\sigma^{\lambda}\ , \tag{6.5.41}$$

and, by (6.5.21) (again),

$$\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta^{\mathbf{X}}_{(\lambda,\kappa)} \in C^\infty(N\backslash G, \omega)\ . \tag{6.5.42}$$

By (6.5.41) and (6.5.22), we have the upper bounds

$$\big(\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta^{\mathbf{X}}_{(\lambda,\kappa)}\big)(g) \ll_{\omega,\theta,\mathbf{X}} (\rho(g))^{1+\sigma} \qquad \text{for } (\lambda,\kappa) \in I(\ell_1, q_1) \text{ and all } g \in G \text{ such that } \rho(g) \leq 1. \tag{6.5.43}$$

Therefore, and by virtue of the identity (6.5.40), it follows that one has also:

$$\big(\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta\big)(g) \ll_{\omega,\theta,\mathbf{X}} (\rho(g))^{1+\sigma} \qquad \text{for } g \in G \text{ such that } \rho(g) \leq 1. \tag{6.5.44}$$

Given it was shown that $\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta \in C^\infty(N\backslash G, \omega)$, and given the results of (6.5.21) and (6.5.22), and the results (6.5.42), (6.5.43) and (6.5.44) just obtained, it follows by Lemma 6.2.2 that the Poincaré series $(P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta)(g)$, $(P^{\mathfrak{a}}\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta)(g)$ and $(P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger}\theta^{\mathbf{X}}_{(\lambda,\kappa)})(g)$ (for all $(\lambda,\kappa) \in I(\ell_1, q_1)$) are each well-defined and absolutely convergent for all $g \in G$. Indeed, by Lemma 6.2.2 and Corollary 6.2.3, the Poincaré series $(P^{\mathfrak{a}}\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta)(g)$ is uniformly convergent in each compact subset of G, and the sum of this series is a function $P^{\mathfrak{a}}\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta : G \to \mathbb{C}$ which is continuous on G. Consequently the proposition of Section 1.72 of [43] (relating to 'term by term' differentiation of a series) implies that if $\mathbf{X} \in \mathfrak{sl}(2,\mathbb{C})$ and $g \in G$ then $(\mathbf{X}P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta)(g)$ exists, and one has:

$$\begin{aligned}
\big(P^{\mathfrak{a}}\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta\big)(g) &= \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma} \big(\mathbf{X}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta\big)\big(g_{\mathfrak{a}}^{-1}\gamma g\big) = \\
&= \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma} \frac{\mathrm{d}}{\mathrm{d}t} \big(\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta\big)\big(g_{\mathfrak{a}}^{-1}\gamma g \exp(t\mathbf{X})\big)\Big|_{t=0} = \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma} \big(\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta\big)\big(g_{\mathfrak{a}}^{-1}\gamma g \exp(t\mathbf{X})\big)\right)\Bigg|_{t=0} = \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta\big)\big(g\exp(t\mathbf{X})\big)\Big)\Big|_{t=0} = \big(\mathbf{X}P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger}\theta\big)(g)\ .
\end{aligned}$$

This proves that (6.5.38) holds if $\mathbf{X} \in \mathfrak{sl}(2,\mathbb{C})$. In the remaining cases, where one has $\mathbf{X} \in \mathfrak{g} - \mathfrak{sl}(2,\mathbb{C})$, the differential operator $\mathbf{X}$ may be expressed as a linear combination (with complex coefficients) of the six elements of the set $\mathcal{B} \subset \mathfrak{sl}(2,\mathbb{C})$ that is defined just below (6.5.26). Therefore, given the linearity inherent in the definition (1.5.4) of Poincaré series, one may deduce these remaining cases of the result in (6.5.38) from those cases of the result that have already been proved. Similarly, given the simple fact (already noted) of the convergence of the relevant Poincaré series, one may deduce that (6.5.40) implies both of the identities stated in (6.5.39) (with, moreover, $\theta^{\mathbf{X}} = (\theta^{\mathbf{X}}_{(\lambda,\kappa)})_{(\lambda,\kappa)\in I(\ell_1,q_1)} \in \prod_{(\lambda,\kappa)\in I(\ell_1,q_1)} \mathcal{T}_\sigma^\lambda$, by virtue of (6.5.41)) ∎

Lemma 6.5.8. *Let $\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta$ be as defined in (6.5.2). Then one has $P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta \in C^\infty(G) \cap L^\infty(\Gamma\backslash G;\ell,q)$ and $P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta| \in C^0(G) \cap L^\infty(\Gamma\backslash G)$.*

Proof. By the relevant definitions in (6.5.2) and (6.5.20), we have

$$\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta = \widetilde{\mathbf{L}}^{\omega,\dagger}_{\ell,q}\eta + b(\omega;\ell,q;\eta)(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(1,0)\ . \tag{6.5.45}$$

One may moreover observe, given the relations in (6.5.4), (6.5.21) and (6.5.22), that it follows by Lemma 6.2.2 and Corollary 6.2.3 that the Poincaré series $(P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta)(g)$, $(P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta|)(g)$ and $(P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,\dagger}_{\ell,q}\eta)(g)$ are each convergent for all $g \in G$, and have sums that are continuous Γ-automorphic functions on G:

$$P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta,\, P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta|,\, P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,\dagger}_{\ell,q}\eta \in C^0(\Gamma\backslash G)\ . \tag{6.5.46}$$

By (6.5.45), (1.5.4) and the convergence just noted, we may deduce that

$$P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta = P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,\dagger}_{\ell,q}\eta + b(\omega;\ell,q;\eta)P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(1,0)\ . \tag{6.5.47}$$

We show next that

$$P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,\dagger}_{\ell,q}\eta \in C^\infty(G) \qquad \text{and} \qquad P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,\dagger}_{\ell,q}\eta \ \text{ is of } K\text{-type } (\ell,q). \tag{6.5.48}$$

The proof is by induction; before coming to it we firstly introduce some relevant terminology. For each non-negative integer j, we define $\mathcal{P}(j)$ to be the proposition that

$$\text{if } \ell_1, q_1 \in \mathbb{Z},\ \ell_1 \geq |q_1| \ \text{and}\ \theta \in \mathcal{T}_\sigma^{\ell_1} \ \text{then}\ P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,\dagger}_{\ell_1,q_1}\theta \in C^j(G)\ , \tag{6.5.49}$$

where $C^j(G)$ denotes the space of all functions $f : G \rightarrow \mathbb{C}$ that satisfy, for each $(\mathbf{X}_1,\ldots,\mathbf{X}_j) \in \mathfrak{g}^j$, the condition that the j-th order derivative $\mathbf{X}_j\mathbf{X}_{j-1}\cdots\mathbf{X}_1 f$ be defined and continuous on G (so that $C^0(G)$ is, as previously, the space of all continuous functions $f : G \rightarrow \mathbb{C}$). Note that it is trivially implicit in the definition of of $C^j(G)$, just given, that $C^j(G) \supseteq C^{j+m}(G)$ for all non-negative integers j and m; less trivially, one has

$$\bigcap_{j=0}^{\infty} C^j(G) = C^\infty(G)\ . \tag{6.5.50}$$

The equality (6.5.50) may be proved by determining local coordinates $x_1(g),\ldots,x_6(g) \in \mathbb{R}$ for G, such that, for $j = 1,\ldots,6$, one has an operator identity of the form

$$\partial/\partial x_j = c_{1,j}(g)\mathbf{H}_1 + c_{2,j}(g)\mathbf{H}_2 + c_{3,j}(g)\mathbf{V}_1 + c_{4,j}(g)\mathbf{V}_2 + c_{5,j}(g)\mathbf{W}_1 + c_{6,j}(g)\mathbf{W}_2\ ,$$

in which $\mathbf{H}_1,\mathbf{H}_2,\mathbf{V}_1,\mathbf{V}_2,\mathbf{W}_1,\mathbf{W}_2$ are the elements of the basis $\mathcal{B}$ of $\mathfrak{sl}(2,\mathbb{C})$ defined below (6.5.26), and the coeffients $c_{1,j}\ldots,c_{6,j}$ are (verifiably) functions in the space $C^\infty(G)$. We omit the (not very enlightening) details of this proof, and merely note that it requires, in addition to the Iwasawa coordinates, some alternative system of coordinates: this is due to singularities which occur in the relevant coefficients when the Iwasawa coordinate θ is an integer multiple of π.

As a starting point for our proof by induction of (6.5.48), we observe that since our only assumption concerning η is that $\eta \in \mathcal{T}_\sigma^\ell$, and since we assume nothing more of ℓ and q than that $\ell, q \in \mathbb{Z}$ and $\ell \geq |q|$, the fact (recorded in (6.5.46)) of our having shown that $P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta \in C^0(\Gamma \backslash G)$ is enough for us to infer that

$$\mathcal{P}(0) \text{ is true.} \tag{6.5.51}$$

Suppose now that J is a non-negative integer such that

$$\mathcal{P}(J) \text{ is true.} \tag{6.5.52}$$

Let $\ell_1, q_1 \in \mathbb{Z}$ satisfy $\ell_1 \geq |q_1|$, let $\theta \in \mathcal{T}_\sigma^{\ell_1}$, and let $\mathbf{X}_1, \ldots, \mathbf{X}_{J+1} \in \mathfrak{g}$. Then, by the results (6.5.38)-(6.5.39) of Lemma 6.5.7, we have

$$\mathbf{X_1} P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger} \theta = \sum_{(\lambda,\kappa) \in I(\ell_1,q_1)} P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger} \theta_{(\lambda,\kappa)}^{\mathbf{X}_1} , \tag{6.5.53}$$

for some $\theta^{\mathbf{X}_1} \in \prod_{(\lambda,\kappa) \in I(\ell_1,q_1)} \mathcal{T}_\sigma^\lambda$. Since it therefore follows by our induction hypothesis (6.5.52) (i.e. by the case $j = J$ of what is stated in (6.5.49)) that every summand on the right-hand side of Equation (6.5.53) is a function in the space $C^J(G)$, we may deduce that the first order derivative $f = \mathbf{X}_1 P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger} \theta$ lies in the same space; it follows that the $(J+1)$-st order derivative $\mathbf{X}_{J+1} \mathbf{X}_J \cdots \mathbf{X}_1 P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger} \theta$ (which is a J-th order derivative of f) is both defined and continuous on G. Since our only assumption concerning $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{J+1}$ is that these are elements of $\mathfrak{g}$, the conclusion just reached is proof that $P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger} \theta$ lies in the space $C^{J+1}(G)$. Therefore, given that our assumptions concerning ℓ_1, q_1 and θ match the conditions stated in (6.5.49), it may (similarly) be inferred that the proposition stated in (6.5.49) is true for $j = J + 1$.

The above discussion (subsequent to (6.5.52)) has shown that for each non-negative integer J the proposition $\mathcal{P}(J)$ implies its successor $\mathcal{P}(J + 1)$. This, combined with the result (6.5.51), implies (by induction) that the proposition $\mathcal{P}(j)$ stated in (6.5.49) is true for all non-negative integers j, and so we may conclude that, subject to the conditions on ℓ_1, q_1 and θ in (6.5.49) being satisfied, one has $P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell_1,q_1}^{\omega,\dagger} \theta \in \bigcap_{j=0}^\infty C^j(G)$. Hence, given (6.5.50), we obtain the first part of what is asserted in (6.5.48).

We begin our verification of the second part of (6.5.48) with the observation that, with $\mathbf{H}_2 \in \mathfrak{sl}(2, \mathbb{C})$ defined as in (1.2.9), it follows by the result (6.5.38) of Lemma 6.5.7, the second part of (6.5.21), and the definitions (1.3.1) and (1.5.4), that

$$\mathbf{H}_2 P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta = P^{\mathfrak{a}} \mathbf{H}_2 \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta = P^{\mathfrak{a}} (-iq) \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta = -iq P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta . \tag{6.5.54}$$

Moreover, given any choice of $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$, the results of Lemma 6.5.7 imply that

$$\mathbf{X} P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta = \sum_{(\lambda,\kappa) \in I(\ell,q)} P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger} \eta_{(\lambda,\kappa)}^{\mathbf{X}} ,$$

where $\eta^{\mathbf{X}} \in \prod_{(\lambda,\kappa) \in I(\ell,q)} \mathcal{T}_\sigma^\lambda$. Two further applications of Lemma 6.5.7 then show that one also has

$$\sum_{(\lambda,\kappa) \in I(\ell,q)} \mathbf{Y} P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger} \eta_{(\lambda,\kappa)}^{\mathbf{X}} = \sum_{(\lambda,\kappa) \in I(\ell,q)} P^{\mathfrak{a}} \mathbf{Y} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger} \eta_{(\lambda,\kappa)}^{\mathbf{X}} = P^{\mathfrak{a}} \mathbf{Y} \sum_{(\lambda,\kappa) \in I(\ell,q)} \widetilde{\mathbf{L}}_{\lambda,\kappa}^{\omega,\dagger} \eta_{(\lambda,\kappa)}^{\mathbf{X}} = P^{\mathfrak{a}} \mathbf{Y} \mathbf{X} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta$$

(the penultimate equality here being a consequence of the linearity of the differential operator $\mathbf{Y}$, and the linearity inherent in the definition (1.5.4) of Poincaré series); this proves that $\mathbf{Y}\mathbf{X} P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta = P^{\mathfrak{a}} \mathbf{Y}\mathbf{X} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta$ for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$, and so, given the first equality of (1.2.11), we obtain (similarly to (6.5.54)) the result that

$$\Omega_{\mathfrak{k}} P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta = P^{\mathfrak{a}} \Omega_{\mathfrak{k}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta = -\frac{1}{2} \left(\ell^2 + \ell \right) P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta . \tag{6.5.55}$$

By (6.5.54) and (6.5.55), the Poincaré series $P^{\mathfrak{a}} \widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger} \eta$ is a function of K-type (ℓ, q). This completes the verification of what is asserted in (6.5.48). Hence, and by Lemma 6.5.2, each of the two Poincaré series

$P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell,q}^{\omega,\dagger}\eta$ and $P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(1,0)$ is a function of K-type (ℓ,q) lying in the space $C^\infty(G)$. Therefore we may deduce from the identity (6.5.47) that

$$P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta \in C^\infty(G) \qquad \text{and} \qquad P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta \quad \text{is of } K\text{-type } (\ell,q). \tag{6.5.56}$$

Given (6.5.56), the results recorded in (6.5.46), and the definitions of the two spaces $L^\infty(\Gamma\backslash G;\ell,q)$ and $L^\infty(\Gamma\backslash G)$ (in, and above, (6.5.6)), it suffices for completion of the proof of the lemma that we show that the Poincaré series $P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta$ and $P^{\mathfrak{a}}|\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta|$ are bounded functions on G. To this end we note that, by (6.5.2), (6.5.1) and the result (6.4.8) of Theorem 6.4.1, one has:

$$\big(\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta\big)(g) = \big(\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta\big)(g) \ll_{\omega,\eta,A} (\rho(g))^{-A} \qquad \text{for all } g\in G \text{ such that } \rho(g)\geq 2, \text{ and all } A\in[0,\infty).$$

We also have (6.5.22), and so it certainly follows that both of the conditions (6.2.5) and (6.2.22) are satisfied when one has (there): $R_0=1$, $\sigma_0=\sigma$, $R_\infty=2$, $\sigma_\infty=10^{10}$ (say), and either $f_\omega=\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta$, or else $\omega=0$ and $f_0=|\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta|$. Since $\sigma\in(1,2)$ (by hypothesis), since $10^{10}\geq 1$, and since the relations stated in (6.5.4) hold, it therefore follows by Corollary 6.2.10 that each of the Poincaré series $P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta$ and $P^{\mathfrak{a}}|\widetilde{\mathbf{L}}_{\ell,q}^{\omega,*}\eta|$ is indeed a bounded function on G ■

Our next lemma generalises the case $\mathfrak{O}=\mathbb{Z}[i]$ of Theorem 7.6.9 of [11] and is a minor addition to the theory of the Linnik-Selberg series

$$Z_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}(s) = \sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}(\omega,\omega';c)}{|c|^{4s}} \qquad (\,\omega'\in\mathfrak{O},\ \mathrm{Re}(s)>3/4). \tag{6.5.57}$$

In Theorem 2.16 of [7] Cogdell, Li, Piatetski-Shapiro and Sarnak have shown that if $\mathfrak{a}$ and $\mathfrak{b}$ are Γ-equivalent cusps then the Linnik-Selberg series $Z_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}(s)$ can be meromorphically continued into all of $\mathbb{C}$. In the case of the analogous series involving generalised Kloosterman sums associated with any group $\Gamma'<SL(2,\mathbb{R})$ that is a congruence subgroup with respect to $SL(2,\mathbb{Z})$, the corresponding meromorphic continuation was obtained by Selberg, in Section 3 of [41].

Lemma 6.5.9. *Let $\omega,\omega'\in\mathfrak{O}$, and let ω be non-zero. Suppose that $\sigma^*>3/4$. Then the series*

$$\sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{|S_{\mathfrak{a},\mathfrak{b}}(\omega,\omega';c)|}{|c|^{4s}} \tag{6.5.58}$$

is uniformly convergent in the half-plane where $\mathrm{Re}(s)\geq\sigma^*$. For $\sigma'\geq\sigma^*$, one has the upper bound

$$\sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{|S_{\mathfrak{a},\mathfrak{b}}(\omega,\omega';c)|}{|c|^{4\sigma'}} \leq 2^{-1/2}|\omega|\,|m_{\mathfrak{a}}m_{\mathfrak{b}}|^{2-2\sigma^*}\left(\sum_{\alpha|q_0}\frac{1}{|\alpha|}\right)^2 \zeta_{\mathbb{Q}(i)}^2(2\sigma^*-1/2)\ , \tag{6.5.59}$$

where $\zeta_{\mathbb{Q}(i)}(s)=(1/4)\sum_{0\neq\alpha\in\mathfrak{O}}|\alpha|^{-2s}$ and $m_{\mathfrak{c}}$ and q_0 are as in (6.1.25)

Proof. Since $|c|^{4s}$ has absolute value $|c|^{4\mathrm{Re}(s)}$ it will certainly suffice to prove that the series is uniformly convergent in the real interval $[\sigma^*,\infty)$. Moreover, by (6.1.25) and the result (6.1.26) of Lemma 6.1.5, the series (6.5.58) is a Dirichlet series of the special type dealt with in Chapter 9 of [43]: in particular, the condition $c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ implies that $|c|^4$ is a positive integer. Hence (given that, for $n\in\mathbb{N}$, the real function $s\mapsto n^{-s}$ is monotonic decreasing), it suffices for proof of the lemma that we show that the series in (6.5.58) is convergent for all $s>3/4$. A quite standard calculation (which we omit) shows that the convergence of this series, for $s>3/4$, may be deduced from the results (6.1.26) and (6.1.27) of Lemma 6.1.5: this same calculation also yields the upper bound in (6.5.59) ■

Lemma 6.5.10. *Let* $\mathrm{Re}(\nu) > 1$, *let* $0 \neq \omega \in \mathfrak{O}$ *and* $\omega' \in \mathfrak{O}$, *and let* $f_\omega = \mathbf{M}_\omega \varphi_{\ell,q}(\nu, 0)$. *Then, for each* $g \in G$, *the integral on the right-hand side of Equation (6.2.13) exists, and the equation (6.2.13) defines a function* $F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega \in C^0(N\backslash G, \omega)$ *satisfying, for* $g \in G$,

$$\begin{aligned}\left(F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega\right)(g) = \zeta_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}\!\left(\frac{1+\nu}{2}\right) \left(\mathbf{J}_{\omega'}\varphi_{\ell,q}(\nu,0)\right)(g) \,+ \\ + \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma \,:\, \gamma\mathfrak{b} = \mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} \in h[u(\gamma)]N}} \delta_{\omega u(\gamma), \omega'/u(\gamma)} \left(\mathbf{M}_\omega \varphi_{\ell,q}(\nu,0)\right) \left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g\right),\end{aligned} \tag{6.5.60}$$

where

$$\zeta_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}(s) = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \mathcal{J}^*_{2s-1,0}\left(\frac{2\pi\sqrt{\omega\omega'}}{c}\right) \frac{S_{\mathfrak{a},\mathfrak{b}}(\omega,\omega';c)}{|c|^{4s}} \tag{6.5.61}$$

(with $\mathcal{J}^*_{\nu,p}(z)$ *as given by Equation (6.3.12) of Lemma 6.3.2). Regarding the case* $\omega' = 0$, *one has moreover:*

$$F_0^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \frac{\sin(\pi\nu)}{\nu^2} \frac{\Gamma(\ell+1-\nu)}{\Gamma(\ell+1+\nu)} Z_{\omega,0}^{\mathfrak{a},\mathfrak{b}}\!\left(\frac{1+\nu}{2}\right) \varphi_{\ell,q}(-\nu,0) = \tag{6.5.62}$$

$$= \pi \frac{\Gamma(\nu)\Gamma(\ell+1-\nu)}{\Gamma(1-\nu)\Gamma(\ell+1+\nu)} \zeta_{\omega,0}^{\mathfrak{a},\mathfrak{b}}\!\left(\frac{1+\nu}{2}\right) \varphi_{\ell,q}(-\nu,0) = \tag{6.5.63}$$

$$= \zeta_{\omega,0}^{\mathfrak{a},\mathfrak{b}}\!\left(\frac{1+\nu}{2}\right) \mathbf{J}_0 \varphi_{\ell,q}(\nu,0) \tag{6.5.64}$$

(with $Z_{\omega,0}^{\mathfrak{a},\mathfrak{b}}(s)$ *as in (6.5.57)). When* $s = (1+\nu)/2$ *the sum on the right-hand side of Equation (6.5.61) is absolutely convergent.*

Proof. By Lemma 6.5.1 we have $P^{\mathfrak{a}} f_\omega \in C^0(\Gamma\backslash G)$; the integral in Equation (6.2.13) therefore exists. Moreover, by the observation preceding (6.3.4), we have $f_\omega \in C^\infty(N\backslash G, \omega)$, and by the estimate (6.3.7) of Lemma 6.3.1 the case $\sigma_0 = \mathrm{Re}(\nu)$, $R_0 = 1$ of the condition (6.2.5) is satisfied. Therefore it follows by Lemma 6.2.5 that the equations (1.5.4) and (6.2.13) define a function $F_{\omega'}^{\mathfrak{b}} P^{\mathfrak{a}} f_\omega \in C^0(N\backslash G, \omega)$, and that a valid formula for this function is given by the case $\mathfrak{a}' = \mathfrak{b}$ of Equation (1.5.5) (in which equation, moreover, all the relevant summations are absolutely convergent).

Assuming now that $g \in G$, the formula (1.5.5) may be applied: we thereby obtain (by firstly recalling that $f_\omega = \mathbf{M}_\omega \varphi_{\ell,q}(\nu,0)$, and then making appropriate use of Equation (6.3.5) and the results (6.3.10)-(6.3.12) of Lemma 6.3.2) both the result (6.5.62) and the case $\omega' \neq 0$ of the result stated in (6.5.60)-(6.5.61). The results (6.5.63) and (6.5.64) follow by virtue of the definitions (6.3.12), (1.9.6) (which imply $\mathcal{J}^*_{\nu,0}(0) = (J^*_\nu(0))^2 = (\Gamma(\nu+1))^{-2}$) and the evaluation of $\mathbf{J}_0\varphi_{\ell,q}(\nu,p)$ in (1.5.18). Given that $\omega \neq 0$, the case $\omega' = 0$ of the result in (6.5.60)-(6.5.61) follows from (6.5.64). The absolute convergence, at $s = (1+\nu)/2$, of the sum in (6.5.61) is a corollary of the point noted, in parenthesis, at the end of the last paragraph ∎

In stating the remaining lemmas of this subsection we shall use (without comment) the same terminology as is used in the statement of Lemma 6.5.10. In particular, $\mathcal{J}^*_{\nu,p}(z)$ denotes the function defined by (6.3.12) and (1.9.6) (or (1.9.8)), and $\zeta_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}(s)$ denotes the function defined by (6.5.61) and (1.5.8)-(1.5.10).

Lemma 6.5.11. *Let* $z \in \mathbb{C}$. *Then the complex function* $\nu \mapsto \mathcal{J}^*_{\nu,0}(z)$ *is entire, and one has*

$$\left|\mathcal{J}^*_{\mu-1/2,0}(z)\right| \leq \frac{\Gamma^2(\mathrm{Re}(\mu))\, e^{2|\mathrm{Im}(z)|}}{\Gamma^2(\mathrm{Re}(\mu+1/2))\, |\Gamma(\mu)|^2} \qquad \text{for } \mathrm{Re}(\mu) > 0. \tag{6.5.65}$$

Proof. For all $\xi \in \mathbb{C}$, the series in Equation (1.9.6) is uniformly convergent in some (or indeed any) compact neighbourhood of ξ. It may therefore be deduced that the equation (1.9.6) defines a complex

function $\nu \mapsto J^*_\nu(z)$ which is entire; given that one may substitute $\overline{z}$ for z in the foregoing, it therefore follows that the complex function $\nu \mapsto J^*_\nu(z) J^*_\nu(\overline{z}) = \mathcal{J}^*_{\nu,0}(z)$ is entire.

Suppose now that $\mathrm{Re}(\mu) > 0$. Then, by (1.9.8) and Poisson's formula, Equation 10.9.4 of [38], one has

$$J^*_{\mu-1/2}(z) = (z/2)^{1/2-\mu} J_{\mu-1/2}(z) = \frac{2}{\pi^{1/2}\Gamma(\mu)} \int_0^1 \left(1-t^2\right)^{\mu-1} \cos(zt)\mathrm{d}t \, .$$

Hence, by using the bound $\max_{t\in[0,1]} |\cos(zt)| \leq \exp(|\mathrm{Im}(z)|)$, the substitution $t^2 = u$, and Euler's evaluation of his Beta-function, $B(m,n)$, one finds that $|J^*_{\mu-1/2}(z)| \leq \exp(|\mathrm{Im}(z)|)|\Gamma(\mu)|^{-1}\Gamma(\mathrm{Re}(\mu))/\Gamma(1/2+\mathrm{Re}(\mu))$. One may substitute $\overline{z}$ for z in the last inequality, and so obtain (by (6.3.12)) the result stated in (6.5.65) ■

Lemma 6.5.12. *Let $0 \neq \omega \in \mathfrak{O}$ and $\omega' \in \mathfrak{O}$. Then the complex function $s \mapsto \zeta^{\mathfrak{a},\mathfrak{b}}_{\omega,\omega'}(s)$ is holomorphic on the half-pane where $\mathrm{Re}(s) > 3/4$, and if $\mathcal{D}$ is a non-empty compact subset of that half-plane then one has*

$$\max_{s\in\mathcal{D}} \left| \zeta^{\mathfrak{a},\mathfrak{b}}_{\omega,\omega'}(s) \right| = O_{\Gamma,\mathcal{D}}\left(|\omega| e^{4\pi\sqrt{|\omega\omega'|}} \right) . \tag{6.5.66}$$

Proof. Let $\mathcal{D}$ be any non-empty compact subset of the half-plane $H = \{s \in \mathbb{C} : \mathrm{Re}(s) > 3/4\}$. Then there is some $\sigma^* = \sigma^*(\mathcal{D}) \in (3/4, \infty)$ and some $r_1 = r_1(\mathcal{D}) \in [\sigma^*, \infty)$ such that one has:

$$|s| \leq r_1 < \infty \qquad \text{and} \qquad \mathrm{Re}(s) \geq \sigma^* > 3/4 \, , \qquad \text{for all } s \in \mathcal{D}. \tag{6.5.67}$$

Suppose moreover that $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$. Then, by (6.1.25) and the result (6.1.26) of Lemma 6.1.5, it follows that $|c| \geq \sqrt{|m_{\mathfrak{a}} m_{\mathfrak{b}}|} \geq 1$. Hence, and since (6.5.67) implies that $\min_{s\in\mathcal{D}} \mathrm{Re}(2s) > 3/2 > 1/2$, the bound (6.5.65) of Lemma 6.5.11 implies that for all $s \in \mathcal{D}$ one has

$$\begin{aligned} \left| \mathcal{J}^*_{2s-1,0}\left(\frac{2\pi\sqrt{\omega\omega'}}{c} \right) \frac{S_{\mathfrak{a},\mathfrak{b}}(\omega,\omega';c)}{|c|^{4s}} \right| &\leq \frac{\Gamma^2(\,\mathrm{Re}(2s-1/2))}{\Gamma^2(\,\mathrm{Re}(2s))\,|\Gamma(2s-1/2)|^2} \exp\left(2 \left| \mathrm{Im}\left(\frac{2\pi\sqrt{\omega\omega'}}{c} \right) \right| \right) \frac{|S_{\mathfrak{a},\mathfrak{b}}(\omega,\omega';c)|}{|c|^{4\mathrm{Re}(s)}} \ll_{r_1} \\ &\ll_{r_1} \exp\left(4\pi\sqrt{|\omega\omega'|} \right) \frac{|S_{\mathfrak{a},\mathfrak{b}}(\omega,\omega';c)|}{|c|^{4\mathrm{Re}(s)}} \end{aligned} \tag{6.5.68}$$

(the last bound following by virtue of the fact that both $\Gamma(w)$ and $1/\Gamma(w)$ are holomorphic on the compact set $\{w \in \mathbb{C} : |w| \leq 2r_1 \text{ and } \mathrm{Re}(w) \geq 1\}$). The bound (6.5.68), in combination with the first part of Lemma 6.5.9, is enough to imply the uniform convergence, for all $s \in \mathcal{D}$, of the series on the right-hand side of Equation (6.5.61). It therefore follows (given the first part of Lemma 6.5.11) that the equation (6.5.61) defines a complex function $s \mapsto \zeta^{\mathfrak{a},\mathfrak{b}}_{\omega,\omega'}(s)$ which is holomorphic on the given compact set $\mathcal{D} \subset H$. From this we may infer, by reason of $H = \{s \in \mathbb{C} : \mathrm{Re}(s) > 3/4\}$ being contained in the union of its compact subsets, that the equation (6.5.61) defines a holomorphic function $\zeta^{\mathfrak{a},\mathfrak{b}}_{\omega,\omega'} : H \to \mathbb{C}$.

Finally, by (6.5.61), (6.5.68), (6.5.67), the upper bound (6.5.59) of Lemma 6.5.9, and the result in (6.1.25), we find that, for all $s \in \mathcal{D}$, one has

$$\zeta^{\mathfrak{a},\mathfrak{b}}_{\omega,\omega'}(s) \ll_{r_1} \exp\left(4\pi\sqrt{|\omega\omega'|} \right) |\omega|\, |m_{\mathfrak{a}} m_{\mathfrak{b}}|^{1/2} \left(\sum_{\alpha | q_0} \frac{1}{|\alpha|} \right)^2 \zeta^2_{\mathbb{Q}(i)}(2\sigma^* - 1/2) \, ,$$

where $|m_{\mathfrak{a}} m_{\mathfrak{b}}| \leq |q_0|^2$. The result (6.5.66) follows from this: for $2\sigma^* - 1/2 > 1$, and we have also that r_1 and σ^* need depend only on $\mathcal{D}$, while (since $\Gamma = \Gamma_0(q_0)$) the group Γ determines the ideal $q_0\mathfrak{O} \subset \mathfrak{O}$ ■

Lemma 6.5.13. *Let $0 \neq \omega \in \mathfrak{O}$. Suppose that $\mathfrak{a} \stackrel{\Gamma}{\sim} \mathfrak{b}$, and let $\gamma \in \Gamma$ be such that $\gamma\mathfrak{b} = \mathfrak{a}$. Then there exists a unique pair $(u,z) = (u(\gamma), z(\gamma)) \in \mathbb{C}^* \times \mathbb{C}$ such that $g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} = h[u]n[z]$; the relevant $u \in \mathbb{C}^*$ is a square root of some unit $\epsilon = \epsilon(\gamma) \in \mathfrak{O}^*$, and one has:*

$$\left(\mathbf{M}_\omega \varphi_{\ell,q}(\nu,0) \right)(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g) = \psi_{\epsilon\omega}(n[z]) \left(\mathbf{M}_{\epsilon\omega} \varphi_{\ell,q}(\nu,0) \right)(g) \qquad (\nu \in \mathbb{C},\, g \in G), \tag{6.5.69}$$

$$\sum_{\omega' \in \mathfrak{O}} \delta_{\omega u, \omega'/u} = \sum_{\omega' \in \mathfrak{O}} \delta_{\epsilon\omega, \omega'} = 1 \tag{6.5.70}$$

and

$$\Gamma_{\mathfrak{a}} \gamma = \{\gamma_1 \in \Gamma : \gamma_1 \mathfrak{b} = \mathfrak{a}\} \ . \tag{6.5.71}$$

Proof. The results preceding that in (6.5.69) are contained in Lemma 4.1; when combined with (6.3.5) (for $p = 0$) and (6.3.3), those results imply the equality in (6.5.69). The result (6.5.70) is self-evident (given that $u^2 = \epsilon \in \mathfrak{O}^*$). To verify the equality in (6.5.71) one need only note the equivalence (for $\gamma_1 \in \Gamma$) of the four binary relations $\gamma_1 \mathfrak{b} = \mathfrak{a}$, $\gamma_1 \gamma^{-1} \mathfrak{a} = \mathfrak{a}$, $\gamma_1 \gamma^{-1} \in \Gamma_{\mathfrak{a}}$ and $\gamma_1 \in \Gamma_{\mathfrak{a}} \gamma$ ■

By Lemma 6.5.12 and Lemma 6.5.3 we may define, for $\mathrm{Re}(\nu) > 1/2$, $g \in G$ and $\omega' \in \mathfrak{O}$, the term

$$\begin{aligned} \phi_{\omega'}(\nu, g) = {} & \zeta_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}\left(\frac{1+\nu}{2}\right) \left(\mathbf{J}_{\omega'} \varphi_{\ell,q}(\nu, 0)\right)(g) \ + \\ & + \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma \ : \ \gamma \mathfrak{b} = \mathfrak{a} \\ g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} \in h[u(\gamma)] N}} \delta_{\omega u(\gamma), \omega'/u(\gamma)} \left(\mathbf{M}_{\omega} \varphi_{\ell,q}(\nu, 0)\right) \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} g\right) \ . \end{aligned} \tag{6.5.72}$$

When $(\nu, g) \in \mathbb{C} \times G$ is such that the series $\sum_{\omega' \in \mathfrak{O}} \phi_{\omega'}(\nu, g)$ is convergent, we denote the sum of that series by $\Phi(\nu, g)$. Both $\phi_{\omega'}(\nu, g)$ and $\Phi(\nu, g)$ are (of course) dependent on Γ, $g_{\mathfrak{a}}$, $g_{\mathfrak{b}}$, ω and the K-type (ℓ, q), but, since these other parameters are (for the purposes of the current discussion) effectively constants, our main concern in the next lemma is with the dependence of $\phi_{\omega'}(\nu, g)$ and $\Phi(\nu, g)$ on the pair $(\nu, g) \in \mathbb{C} \times G$.

Lemma 6.5.14. *Let $0 \neq \omega \in \mathfrak{O}$. Then, when $r_0, \sigma_2, t_1 \in (0, \infty)$, $\sigma_1 \in (1/2, \sigma_2)$, $G(r_0) = \{g \in G : \rho(g) \geq r_0\}$ and $\mathcal{R} = \mathcal{R}(\sigma_1, \sigma_2, t_1) = \{\nu \in \mathbb{C} : \sigma_1 \leq \mathrm{Re}(\nu) \leq \sigma_2$ and $|\mathrm{Im}(\nu)| \leq t_1\}$, the series $\sum_{\omega' \in \mathfrak{O}} |\phi_{\omega'}(\nu, g)|$ is uniformly convergent for all $(\nu, g) \in \mathcal{R} \times G(r_0)$, and its sum $\Phi(\nu, g)$ satisfies*

$$\Phi(\nu, g) - \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma \\ \gamma \mathfrak{b} = \mathfrak{a}}} \left(\mathbf{M}_{\omega} \varphi_{\ell,q}(\nu, 0)\right) \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} g\right) \ll_{\Gamma,\omega,\ell,\mathcal{R},r_0} \ (\rho(g))^{1 - \mathrm{Re}(\nu)} \ , \tag{6.5.73}$$

for $(\nu, g) \in \mathcal{R} \times G(r_0)$. In particular, the series $\sum_{\omega' \in \mathfrak{O}} \phi_{\omega'}(\nu, g)$ is absolutely convergent for all $(\nu, g) \in \mathbb{C} \times G$ such that $\mathrm{Re}(\nu) > 1/2$. Properties of its sum $\Phi(\nu, g)$ are:

when $g \in G$, the function $\nu \mapsto \Phi(\nu, g)$ is holomorphic for $\mathrm{Re}(\nu) > 1/2$; (6.5.74)

the function $(\nu, g) \mapsto \Phi(\nu, g)$ is continuous on $\{\nu \in \mathbb{C} : \mathrm{Re}(\nu) > 1/2\} \times G$; (6.5.75)

when $\mathrm{Re}(\nu) > 1$, one has $\Phi(\nu, g) = \left(P^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\ell,q}(\nu, 0)\right)(g_{\mathfrak{b}} g)$ for all $g \in G$. (6.5.76)

Proof. By Lemma 4.2, we have $[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \in \{2, 4\}$, so that (by (6.5.71)) the sum over 'γ' in Equation (6.5.72) is always finite. That sum is, moreover, empty unless $\omega' \sim \omega$: for it follows from Lemma 6.5.13 that $u(\gamma)$, in (6.5.72), is always a square root of a unit of $\mathfrak{O}$. Therefore, assuming now that σ_1, σ_2, r_0, t_1, $\mathcal{R}$ and $G(r_0)$, are as specified in the lemma, and that

$$\phi_{\omega'}^*(\nu, g) = \zeta_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}\left(\frac{1+\nu}{2}\right) \left(\mathbf{J}_{\omega'} \varphi_{\ell,q}(\nu, 0)\right)(g) \qquad (\text{for } \mathrm{Re}(\nu) > 1/2,\ g \in G \text{ and } \omega' \in \mathfrak{O}),$$

it will suffice for proof of the first result of the lemma that we show the series $\sum_{0 \neq \omega' \in \mathfrak{O}} |\phi_{\omega'}^*(\nu, g)|$ to be uniformly convergent for $(\nu, g) \in \mathcal{R} \times G(r_0)$.

Since the conditions $\mathrm{Re}(\nu) \geq \sigma_1 > 1/2$ imply $\mathrm{Re}((\nu + 1)/2)) > 3/4$, it follows by Lemma 6.5.12 that

$$\zeta_{\omega,\omega'}^{\mathfrak{a},\mathfrak{b}}\left(\frac{1+\nu}{2}\right) \ll_{\Gamma,\mathcal{R}} \ |\omega| e^{4\pi\sqrt{|\omega\omega'|}} \qquad \text{for } \ \nu \in \mathcal{R},\ \omega' \in \mathfrak{O}. \tag{6.5.77}$$

By the estimate (6.5.14) of Lemma 6.5.3, we have also the upper bound

$$\left(\mathbf{J}_{\omega'}\varphi_{\ell,q}(\nu,0)\right)(g) \ll_{\ell,\sigma_2,r_0} |\Gamma(\nu+1)|^{-1}\,|\omega'|^{\ell+\mathrm{Re}(\nu)}\,(\rho(g))^{\ell+1}\,e^{-2\pi|\omega'|\rho(g)}\ ,$$

for $0 \neq \omega' \in \mathfrak{O}$ and $(\nu, g) \in \mathcal{R} \times G(r_0)$: we rely here on the fact the conditions imply the lower bound $|\omega'| \geq 1$, and so ensure that $|\omega'|\rho(g) \geq r_0$ when $g \in G(r_0)$. Given the definition of the compact set $\mathcal{R} \subset \mathbb{C}$, it follows from this last bound that, since $1/\Gamma(w)$ is an entire complex function, one has:

$$\left(\mathbf{J}_{\omega'}\varphi_{\ell,q}(\nu,0)\right)(g) \ll_{\ell,\mathcal{R},r_0} |\omega'|^{\ell+\sigma_2}\,(\rho(g))^{\ell+1}\,e^{-2\pi|\omega'|\rho(g)} \qquad \text{for } 0 \neq \omega' \in \mathfrak{O},\ (\nu,g) \in \mathcal{R} \times G(r_0). \tag{6.5.78}$$

Combining the bounds in (6.5.77) and (6.5.78) we find that, if $0 \neq \omega' \in \mathfrak{O}$ and $(\nu, g) \in \mathcal{R} \times G(r_0)$, then

$$\begin{aligned}\phi^*_{\omega'}(\nu,g) &\ll_{\Gamma,\ell,\mathcal{R},r_0} |\omega|\,|\omega'|^{\ell+\sigma_2}\,(\rho(g))^{\ell+1}\,e^{-\pi\left(2\rho(g)|\omega'|-4\sqrt{|\omega\omega'|}\right)} \leq \\ &\leq |\omega/\omega'|\,e^{4\pi|\omega|/r_0}r_0^{-\sigma_2}\,(\rho(g)\,|\omega'|)^{\ell+1+\sigma_2}\,e^{-\pi\rho(g)|\omega'|} \ll_{\ell,\omega,\sigma_2,r_0} \\ &\ll_{\ell,\omega,\sigma_2,r_0} e^{-(\pi/2)\rho(g)|\omega'|} \leq e^{-(\pi/2)r_0|\omega'|} \leq \frac{48}{\pi^3 r_0^3|\omega'|^3}\ .\end{aligned} \tag{6.5.79}$$

From this follows (by virtue of the convergence of the series $\sum_{0\neq\omega'\in\mathfrak{O}}|\omega'|^{-3}$) the uniform convergence, for $(\nu,g) \in \mathcal{R}\times G(r_0)$, of the series $\sum_{0\neq\omega'\in\mathfrak{O}}|\phi^*_{\omega'}(\nu,g)|$. By earlier remarks, the first result of the lemma follows.

We prove next the estimate (6.5.73). By (6.5.79) it is seen that, when $(\nu,g) \in \mathcal{R}\times G(r_0)$, one has:

$$\begin{aligned}\sum_{0\neq\omega'\in\mathfrak{O}} \phi^*_{\omega'}(\nu,g) &\ll_{\Gamma,\ell,\omega,\mathcal{R},r_0} \sum_{0\neq\omega'\in\mathfrak{O}} e^{-(\pi/2)\rho(g)|\omega'|} \leq \\ &\leq e^{-(\pi/3)\rho(g)} \sum_{0\neq\omega'\in\mathfrak{O}} \left(\frac{4!}{r_0^4\,|\omega'|^4}\right)^{\pi/6} \ll_{r_0} e^{-(\pi/3)\rho(g)} \leq e^{-\rho(g)}\ .\end{aligned}$$

By (6.5.77), (1.5.18) and (1.3.2) one has, moreover,

$$\phi^*_0(\nu,g) = \zeta^{\mathfrak{a},\mathfrak{b}}_{\omega,0}\!\left(\frac{1+\nu}{2}\right)\varphi_{\ell,q}(-\nu,0)(g)\frac{\pi}{\nu}\prod_{m=1}^{\ell}\left(\frac{m-\nu}{m+\nu}\right) \ll_{\Gamma,\ell,\mathcal{R}} |\omega|\,(\rho(g))^{1-\mathrm{Re}(\nu)}\ ,$$

when $(\nu,g) \in \mathcal{R}\times G(r_0)$. We therefore find that

$$\sum_{\omega'\in\mathfrak{O}} \phi^*_{\omega'}(\nu,g) \ll_{\Gamma,\ell,\omega,\mathcal{R},r_0} (\rho(g))^{1-\mathrm{Re}(\nu)} + e^{-\rho(g)} \ll_{\sigma_2,r_0} (\rho(g))^{1-\mathrm{Re}(\nu)} \qquad \text{for } (\nu,g) \in \mathcal{R}\times G(r_0),$$

and from this the result (6.5.73) follows: for, by (6.5.72) and result (6.5.70) of Lemma 6.5.13, it follows (certainly for all $(\nu,g) \in \mathcal{R}\times G(r_0)$) that one has

$$\Phi(\nu,g) - \sum_{\omega'\in\mathfrak{O}} \phi^*_{\omega'}(\nu,g) = \sum_{\omega'\in\mathfrak{O}} \left(\phi_{\omega'}(\nu,g) - \phi^*_{\omega'}(\nu,g)\right) = \frac{1}{[\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]} \sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma \\ \gamma\mathfrak{b}=\mathfrak{a}}} \left(\mathbf{M}_{\omega}\varphi_{\ell,q}(\nu,0)\right)\left(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} g\right)\ .$$

The first two results of the lemma (stated in and above (6.5.73)) have now been proved, and so, in proving the other results of the lemma, we may freely apply those first two results (in their full generality). In particular, the third result of the lemma (absolute convergence of the series $\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu,g)$ when $(\nu,g) \in \mathbb{C}\times G$ and $\mathrm{Re}(\nu) > 1/2$) is an immediate corollary of the case $\sigma_1 = \mathrm{Re}(\nu)$, $\sigma_2 = \mathrm{Re}(\nu+1)$, $t_1 = |\mathrm{Im}(\nu)|+1$, $r_0 = \rho(g)$ of the uniform convergence noted in the first result of the lemma.

Assume henceforth that $(\nu_0,g_0) \in \mathbb{C}\times G$, and that $\mathrm{Re}(\nu_0) > 1/2$. Given the definition (6.5.72), it follows by (1.5.18), Lemma 6.5.3 and Lemma 6.5.12 that each term of the series $\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu,g_0)$ is holomorphic (as a function of the complex variable ν) in the half-plane where $\mathrm{Re}(\nu) > 1/2$. Therefore (as an application of the first result of the lemma) we may deduce from the uniform convergence of the series $\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu,g)$ for

$(\nu,g)\in\mathcal{R}(\mathrm{Re}(\nu_0/2)+1/4,\mathrm{Re}(2\nu_0),|\mathrm{Im}(\nu_0)|+1)\times\{g_0\}\subset\mathcal{R}(\mathrm{Re}(\nu_0/2)+1/4,\mathrm{Re}(2\nu_0),|\mathrm{Im}(\nu_0)|+1)\times G(\rho(g_0))$ that the function $\nu\mapsto\Phi(\nu,g_0)=\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu,g_0)$ is holomorphic in a neighbourhood of the point $\nu_0\in\mathbb{C}$. This proves the result (6.5.74): for we have assumed nothing more than that $\mathrm{Re}(\nu_0)>1/2$ and $g_0\in G$.

Considering now the series $\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu_0,g)$, we observe that, given the definition (6.5.72), it follows by (1.5.15) and the observation preceding (6.3.4) that each term of this series is a continuous (as a function of g) on all of G. Therefore we may deduce from the uniform convergence of the series $\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu,g)$ for $(\nu,g)\in\{\nu_0\}\times G(\rho(g_0)/2)\subset\mathcal{R}(\mathrm{Re}(\nu_0),\mathrm{Re}(\nu_0)+1,|\mathrm{Im}(\nu_0)|+1)\times G(\rho(g_0)/2)$ (which follows by the first result of the lemma) that the function $g\mapsto\Phi(\nu_0,g)=\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu_0,g)$ is continuous at the point $g_0\in G$. We may infer from this that, when $\mathrm{Re}(\nu)>1/2$, the function $g\mapsto\Phi(\nu,g)$ is continuous on G. This advances us one step towards a proof of the result (6.5.75).

A locally uniform upper bound for $\Phi(\nu,g)$ suffices to complete the proof of (6.5.75). We begin our proof of such a bound by putting: $\sigma_1=\mathrm{Re}(\nu_0/3)+1/3$, $\sigma_2=\mathrm{Re}(3\nu_0)$, $t_1=|\mathrm{Im}(\nu_0)|+2$, $r_0=\rho(g_0)/2$, $r_1=2\rho(g_0)$, $\mathcal{R}=\mathcal{R}(\sigma_1,\sigma_2,t_1)=\{\nu\in\mathbb{C}:\sigma_1\le\mathrm{Re}(\nu)\le\sigma_2\text{ and }|\mathrm{Im}(\nu)|\le t_1\}$ and $G(r_0,r_1)=\{g\in G:r_0\le\rho(g)\le r_1\}$. By the result (6.5.73) (proved earlier), the results of Lemma 6.5.13 (excluding (6.5.70)) and the estimate (6.3.9) of Lemma 6.3.1, we find that if $(\nu,g)\in\mathcal{R}\times G(r_0,r_1)$ then, given the definition of the set $\mathcal{R}=\mathcal{R}(\sigma_1,\sigma_2,t_1)\subset\mathbb{C}$, one has

$$|\Phi(\nu,g)|\le\max_{\epsilon\in\mathfrak{O}^*}\left|\big(\mathbf{M}_{\epsilon\omega}\varphi_{\ell,q}(\nu,0)\big)(g)\right|+O_{\Gamma,\omega,\ell,\mathcal{R},r_0}\left((\rho(g))^{1-\mathrm{Re}(\nu)}\right)=$$
$$=O_{\omega,\ell,r_1,\sigma_2}\left((\rho(g))^{1+\mathrm{Re}(\nu)}\left(1+|\mathrm{Im}(\nu)|\right)^{-2\mathrm{Re}(\nu)-1}e^{\pi|\mathrm{Im}(\nu)|}\right)+O_{\Gamma,\omega,\ell,\mathcal{R},r_0}\left((\rho(g))^{1-\mathrm{Re}(\nu)}\right),$$

$$0<(\rho(g))^{1+\mathrm{Re}(\nu)}\left(1+|\mathrm{Im}(\nu)|\right)^{-2\mathrm{Re}(\nu)-1}e^{\pi|\mathrm{Im}(\nu)|}\le\exp\left(\left(1+\sigma_2\right)r_1+\pi t_1\right)$$

and

$$0<(\rho(g))^{1-\mathrm{Re}(\nu)}\le r_1\exp\left(\sigma_2/r_0\right).$$

One sees from this that the function $(\nu,g)\mapsto\Phi(\nu,g)$ is bounded on $\mathcal{R}(\sigma_1,\sigma_2,t_1)\times G(r_0,r_1)$. It therefore follows, by an application of Cauchy's integral formula for the derivative of a holomorphic function, that the function $(\nu,g)\mapsto(\partial/\partial\nu)\Phi(\nu,g)$ is bounded on the set $\mathcal{R}'\times G(r_0,r_1)$, where $\mathcal{R}'=\mathcal{R}(2\sigma_1-1/2,2\sigma_2/3,t_1-1)$ $=\{\nu\in\mathbb{C}:\mathrm{Re}(2\nu_0/3)+1/6\le\mathrm{Re}(\nu)\le\mathrm{Re}(2\nu_0)\text{ and }|\mathrm{Im}(\nu)|\le|\mathrm{Im}(\nu_0)|+1\}$; consequently there exists some $M\in(0,\infty)$ such that, for all $\nu\in\mathcal{R}'$ and all $g\in G(r_0,r_1)$, one has $|\Phi(\nu,g)-\Phi(\nu_0,g)|\le M|\nu-\nu_0|$ (it should be noted here that, since $\mathrm{Re}(\nu_0)>1/2$, the definitions of $G(r_0,r_1)$ and $\mathcal{R}'$ ensure that the set $\mathcal{R}'\times G(r_0,r_1)$ contains a neighbourhood of the point $(\nu_0,g_0)\in\mathbb{C}\times G$). Hence, by the triangle inequality,

$$|\Phi(\nu,g)-\Phi(\nu_0,g_0)|\le M|\nu-\nu_0|+|\Phi(\nu_0,g)-\Phi(\nu_0,g_0)|\qquad\text{for }(\nu,g)\in\mathcal{R}'\times G(r_0,r_1).\tag{6.5.80}$$

Suppose now that $\varepsilon>0$. Since we assume that $\mathrm{Re}(\nu_0)>1/2$, we know (by work done above) that the function $g\mapsto\Phi(\nu_0,g)$ is continuous on G. Hence there exists a neighbourhood U_0 of g_0 such that $|\Phi(\nu_0,g)-\Phi(\nu_0,g_0)|<\varepsilon/2$ for all $g\in U_0$; we may assume, moreover, that $U_0\subset G(r_0,r_1)$ (if necessary we replace U_0 by $\mathrm{Int}(G(r_0,r_1))\cap U_0$). By this observation, combined with (6.5.80), we find that $|\Phi(\nu,g)-\Phi(\nu_0,g_0)|<\varepsilon$ for all (ν,g) lying in the neighbourhood $\{\nu\in\mathrm{Int}(\mathcal{R}'):|\nu-\nu_0|<\varepsilon/(2M)\}\times U_0$ of the point (ν_0,g_0); since our only assumptions concerning $\nu_0\in\mathbb{C}$, $g_0\in G$ and $\varepsilon\in\mathbb{R}$ are that we have $\mathrm{Re}(\nu_0)>1/2$ and $\varepsilon>0$, this completes the proof of the result (6.5.75).

We now aim to complete the proof of the lemma, by proving the result (6.5.76). Accordingly, it is to be assumed henceforth that we have $\mathrm{Re}(\nu)>1$ and $g\in G$. Since $\mathrm{Re}(\nu)>1$, it follows by the result (6.5.60) of Lemma 6.5.10, combined with the definitions (6.5.72), (6.2.13), (1.4.3), (1.1.21), (1.1.10) and (1.1.3), that for $(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}$ and $\omega'=-n_1+in_2$ we have:

$$\phi_{-n_1+in_2}(\nu,g)=\big(F^{\mathfrak{b}}_{\omega'}P^{\mathfrak{a}}\mathbf{M}_{\omega}\varphi_{\ell,q}(\nu,0)\big)(g)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(x_1,x_2\right)\mathrm{e}\left(n_1x_1+n_2x_2\right)\mathrm{d}x_1\,\mathrm{d}x_2\;,$$

where the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is defined by:

$$f\left(x_1,x_2\right)=\begin{cases}\big(P^{\mathfrak{a}}\mathbf{M}_{\omega}\varphi_{\ell,q}(\nu,0)\big)(g_{\mathfrak{b}}n[x_1+ix_2]g) & \text{if }-1/2\le x_1,x_2<1/2;\\ 0 & \text{otherwise.}\end{cases}\tag{6.5.81}$$

We now seek to apply the two-variable case of Bochner's theorem, Theorem 67 of [2], on Poisson summation in several variables. By the first result of the lemma (proved in the paragraph containing (6.5.79)), we know that the series $\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu,g)$ is absolutely convergent, and so 'Assumption (c)' of Bochner's theorem is satisfied. The function f defined in (6.5.81) may be shown also to satisfy the other hypotheses of the case $k=2$ of Bochner's theorem (i.e. his 'Assumptions (a) and (b)'). Indeed, with regard to Assumption (a) of Theorem 67 of [2] we need only note that, since $\mathrm{Re}(\nu)>1$, it follows by Lemma 6.5.1 that $P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ is a continuous function on the topological group G; with regard to Assumption (b) of Theorem 67 of [2] it is enough to observe (as a consequence of (6.5.81)) that if $-1/2\le x,y<1/2$ then $f(x+m,y+n)=0$ for all $(m,n)\in\mathbb{Z}\times\mathbb{Z}-\{(0,0)\}$.

Bochner's theorem justifies the application of the two variable form of Poisson's summation formula; in light of the point just observed in connection with his 'Assumption (b)', we therefore find that

$$\sum_{n_1=-\infty}^{\infty}\ \sum_{n_2=-\infty}^{\infty}\phi_{-n_1+in_2}(\nu,g)=\sum_{m_1=-\infty}^{\infty}\ \sum_{m_2=-\infty}^{\infty}f\left(m_1,m_2\right)=f(0,0)\ .$$

The sum over n_1 and n_2 here is $\Phi(\nu,g)$, and so (by (6.5.81) for $x_1=x_2=0$) the result (6.5.76) follows ∎

The results (6.5.76) and (6.5.74) of Lemma 6.5.14 show that the sum $\Phi(\nu,g_{\mathfrak{b}}^{-1}g)=\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}(\nu,g_{\mathfrak{b}}^{-1}g)$ provides a means of extending the domain of the function $(\nu,g)\mapsto(P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$ that is 'natural' (in that the extended function is determined by a process of analytic continuation). In referring to the extended function one might avoid the use of any new terminology. However, in the interest of clarity, we choose to let $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ denote, when $\mathrm{Re}(\nu)>1/2$, the function on G satisfying

$$\left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g)=\Phi\left(\nu,g_{\mathfrak{b}}^{-1}g\right)=\sum_{\omega'\in\mathfrak{O}}\phi_{\omega'}\left(\nu,g_{\mathfrak{b}}^{-1}g\right)\qquad(g\in G),\tag{6.5.82}$$

where the terms of the sum over $\omega'\in\mathfrak{O}$ are as indicated in Equation (6.5.72). Note that each function $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ defined in this way is independent of the particular choice of cusp $\mathfrak{b}$ and scaling matrix $g_{\mathfrak{b}}$: for it follows by (6.5.82) and the results (6.5.76) and (6.5.74) of Lemma 6.5.14 that

$$\left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g)=\left(P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g)\qquad\text{for }(\nu,g)\in\mathbb{C}\times G\text{ such that }\mathrm{Re}(\nu)>1,\tag{6.5.83}$$

and that, for each $g\in G$, the function $\nu\mapsto\left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g)$ is holomorphic for $\mathrm{Re}(\nu)>1/2$, and so is completely determined (as a function with domain $\{\nu\in\mathbb{C}:\mathrm{Re}(\nu)>1/2\}$) by the data in (6.5.83).

The new terminology just introduced aids in the clarification of certain steps in subsequent proofs. Although $\mathfrak{b}$, $g_{\mathfrak{b}}$ and the K-type (ℓ,q) are fixed (for the purposes of the present discussion), we nevertheless take (6.5.82) to imply corresponding definitions of $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\lambda,\kappa}(\nu,0)$ for all $\kappa,\lambda\in\mathbb{Z}$ such that $\lambda\ge|\kappa|$, and we allow substitution of other pairs $\mathfrak{c}$, $g_{\mathfrak{c}}$ in place of the pair $\mathfrak{b}$, $g_{\mathfrak{b}}$ (so long as the conditions (1.1.16) and (1.1.20) are satisfied); this obviously has to be accompanied by matching substitutions on the right-hand side of Equation (6.5.72), where the term $\phi_{\omega'}(\nu,g)$ is defined; in order to make this quite clear, we let $\phi_{\omega'}^{\mathfrak{c}}(\lambda,\kappa;\nu,g)$ denote the term $\phi_{\omega'}(\nu,g)$ which Equation (6.5.72) would define were $\mathfrak{c}$, $g_{\mathfrak{c}}$ and the K-type (λ,κ) substituted for $\mathfrak{b}$, $g_{\mathfrak{b}}$ and the K-type (ℓ,q), respectively, and we also put $\Phi^{\mathfrak{c}}(\lambda,\kappa;\nu,g)=\sum_{\omega'\neq0}\phi_{\omega'}^{\mathfrak{c}}(\lambda,\kappa;\nu,g)$.

By the case $\mathfrak{b}=\infty$, $g_{\mathfrak{b}}=h[1]$ of (6.5.82), (6.5.72), Lemma 6.5.13 and Lemma 6.5.14, it follows that, when $0\neq\omega\in\mathfrak{O}$, $\lambda,\kappa\in\mathbb{Z}$, $\lambda\ge|\kappa|$, $g\in G$ and $\mathrm{Re}(\nu)>1/2$, we have:

$$\begin{aligned}\left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\lambda,\kappa}(\nu,0)\right)(g)&=\frac{1}{[\Gamma_{\mathfrak{a}}:\Gamma_{\mathfrak{a}}']}\sum_{\substack{\gamma\in\Gamma_{\mathfrak{a}}'\backslash\Gamma\\ \gamma\infty=\mathfrak{a}}}\left(\mathbf{M}_\omega\varphi_{\lambda,\kappa}(\nu,0)\right)\left(g_{\mathfrak{a}}^{-1}\gamma g\right)+\\&\qquad+\sum_{\omega'\in\mathfrak{O}}\zeta_{\omega,\omega'}^{\mathfrak{a},\infty}\!\left(\frac{1+\nu}{2}\right)\left(\mathbf{J}_{\omega'}\varphi_{\lambda,\kappa}(\nu,0)\right)(g)\ .\end{aligned}\tag{6.5.84}$$

We remark that the sum over $\gamma\in\Gamma_{\mathfrak{a}}'\backslash\Gamma$ in Equation (6.5.84) is evidently empty unless the cusp $\mathfrak{a}$ is Γ-equivalent to the cusp ∞; by Lemma 4.1, Lemma 4.2, the equations (6.3.3) and (6.3.5) and the definition

(1.4.3), it follows moreover that if $\mathfrak{a} \stackrel{\Gamma}{\sim} \infty$ then there exists some $\epsilon \in \mathfrak{O}^*$ and some $\beta \in \mathbb{C}$ (both depending only upon Γ and $g_{\mathfrak{a}}$) such that

$$\frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma \\ \gamma\infty = \mathfrak{a}}} \left(\mathbf{M}_{\omega} \varphi_{\lambda,\kappa}(\nu, 0) \right) \left(g_{\mathfrak{a}}^{-1} \gamma g \right) = \mathrm{e}(\mathrm{Re}(\beta \epsilon \omega)) \sum_{\alpha = \pm 1} \frac{1}{2} \left(\mathbf{M}_{\alpha\epsilon\omega} \varphi_{\lambda,\kappa}(\nu, 0) \right)(g) \ ,$$

when the conditions attached to (6.5.84) are satisfied.

Lemma 6.5.15. *The equation (6.5.82) defines, for* $\mathrm{Re}(\nu) > 1/2$*, a function* $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\ell,q}(\nu, 0) : G \to \mathbb{C}$ *which lies in the space* $C^{\infty}(\Gamma \backslash G)$ *and is of* K*-type* (ℓ, q)*.*

Proof. To avoid both ambiguity and unnecessary repetition we make it our rule that, when (in the course of this proof) there is any application made of some previous result (or definition) of this subsection, it is to be understood that the application in question is made in respect of the special case $\mathfrak{b} = \infty$, $g_{\mathfrak{b}} = h[1]$.

Let $\nu_0 \in \mathbb{C}$ be such that $\mathrm{Re}(\nu_0) > 1/2$. Then, by Lemma 6.5.14, the equation (6.5.82) defines a complex valued function $g \mapsto (\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\ell,q}(\nu_0, 0))(g)$ with domain G. By Lemma 6.5.1, and the results (6.5.76) and (6.5.74) of Lemma 6.5.14, it may moreover be deduced that this function $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\ell,q}(\nu_0, 0) : G \to \mathbb{C}$ is Γ-automorphic (i.e. we obtain this not only when $\mathrm{Re}(\nu_0) > 1$, but also when $1/2 < \mathrm{Re}(\nu_0) \leq 1$).

Suppose that $\mathbf{X} \in \mathcal{B}_1 = \{\mathbf{H}_1, \mathbf{H}_2, \mathbf{F}^+, \mathbf{F}^-, \mathbf{E}^+, \mathbf{E}^-\}$ (the basis of $\mathfrak{g}$ utilised in the proof of Lemma 6.5.5). Since the Jacquet operator $\mathbf{J}_{\omega}$ and Goodman-Wallach operator $\mathbf{M}_{\omega}$ are both linear operators that commute with all elements of $\mathcal{U}(\mathfrak{g})$, and since all elements of $\mathcal{U}(\mathfrak{g})$ act as left-invariant differential operators on the space $C^{\infty}(G)$, it therefore follows from the definition (6.5.72) and the equation (6.5.29) (together with the remarks following it) that, when $\omega' \in \mathfrak{O}$, we have:

$$\mathbf{X}\phi_{\omega'}(\nu_0, g) = \mathbf{X}\phi^{\infty}_{\omega'}(\ell, q; \nu_0, g) = \sum_{(\lambda,\kappa) \in I(\ell,q)} c^{\mathbf{X}}_{\ell,q}(\lambda, \kappa; \nu_0, 0)\, \phi^{\infty}_{\omega'}(\lambda, \kappa; \nu_0, g) \qquad (g \in G), \qquad (6.5.85)$$

where $I(\ell, q)$ is the finite subset of $\mathbb{Z} \times \mathbb{Z}$ defined in (6.5.23), while each coefficient $c^{\mathbf{X}}_{\ell,q}(\lambda, \kappa; \nu_0, 0)$ is a certain polynomial function of ν_0 (and is of course independent of the variable g). By Lemma 6.5.14, it is moreover the case that, for each pair $(\lambda, \kappa) \in I(\ell, q)$, the series $\sum_{\omega' \in \mathfrak{O}} \phi^{\infty}_{\omega'}(\lambda, \kappa; \nu_0, g)$ converges uniformly on any given compact subset of G; by this and the equation (6.5.85), we may deduce that the series $\sum_{\omega' \in \mathfrak{O}} \mathbf{X}\phi^{\infty}_{\omega'}(\ell, q; \nu_0, g)$ is (likewise) uniformly convergent on any given compact subset of G. Moreover, by (6.5.85), (6.5.72) (with the K-type (λ, κ) substituted for the K-type (ℓ, q)), the relation (1.5.15), and the remark preceding (6.3.4), one sees that each term $\mathbf{X}\phi^{\infty}_{\omega'}(\ell, q; \nu_0, g)$ in the latter series is a continuous function of the variable g. It therefore follows by the proposition in Section 1.72 of [43] that for $g \in G$ the derivative $\mathbf{X}\Phi^{\infty}(\ell, q; \nu_0, g) = \mathbf{X}(\sum_{\omega' \in \mathfrak{O}} \phi^{\infty}_{\omega'}(\ell, q; \nu_0, g))$ exists, and is equal to $\sum_{\omega' \in \mathfrak{O}} \mathbf{X}\phi^{\infty}_{\omega'}(\ell, q; \nu_0, g)$: note that, although we omit to give a detailed justification of the reasoning here, the details omitted are similar to details provided in our proof of the result (6.5.38) of Lemma 6.5.7.

Now, by the definition (6.5.82), it follows that $\mathbf{X}(\sum_{\omega' \in \mathfrak{O}} \phi^{\infty}_{\omega'}(\ell, q; \nu_0, g)) = (\mathbf{X}\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\ell,q}(\nu_0, 0))(g)$ for all $g \in G$. On the other hand, by (6.5.85), (6.5.23) and (6.5.82) (the last applied, this time, with the K-type (λ, κ) substituted for (ℓ, q)), we have, for $g \in G$,

$$\begin{aligned} \sum_{\omega' \in \mathfrak{O}} \mathbf{X}\phi^{\infty}_{\omega'}(\ell, q; \nu_0, g) &= \sum_{(\lambda,\kappa) \in I(\ell,q)} c^{\mathbf{X}}_{\ell,q}(\lambda, \kappa; \nu_0, 0) \sum_{\omega' \in \mathfrak{O}} \phi^{\infty}_{\omega'}(\lambda, \kappa; \nu_0, g) = \\ &= \sum_{(\lambda,\kappa) \in I(\ell,q)} c^{\mathbf{X}}_{\ell,q}(\lambda, \kappa; \nu_0, 0) \left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\lambda,\kappa}(\nu_0, 0) \right)(g) \ , \end{aligned}$$

where, by (6.5.82) and the result (6.5.75) of Lemma 6.5.14, the relevant functions $g \mapsto (\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\lambda,\kappa}(\nu_0, 0))(g)$ are each continuous on G. The convergence of the relevant series here is not at issue (see the previous paragraph). Given the identity $\mathbf{X}(\sum_{\omega' \in \mathfrak{O}} \phi^{\infty}_{\omega'}(\ell, q; \nu_0, g)) = \sum_{\omega' \in \mathfrak{O}} \mathbf{X}\phi^{\infty}_{\omega'}(\ell, q; \nu_0, g)$ obtained in the previous paragraph, and the points just noted in this paragraph, we find that the function $g \mapsto (\mathbf{X}\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\ell,q}(\nu_0, 0))(g)$ is defined and continuous on G, and that one has:

$$\mathbf{X}\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\ell,q}(\nu_0, 0) = \sum_{(\lambda,\kappa) \in I(\ell,q)} c^{\mathbf{X}}_{\ell,q}(\lambda, \kappa; \nu_0, 0)\, \underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega} \varphi_{\lambda,\kappa}(\nu_0, 0) \ . \qquad (6.5.86)$$

Since the definition (6.5.23) implies that the set $I(\ell,q)$ is finite, and since $\mathcal{B}_1$ is a $\mathbb{C}$-basis of $\mathfrak{g}$, it may be shown, through the iterative application of Equation (6.5.86) (with the choice of $\mathbf{X}\in\mathfrak{g}$ and K-type (ℓ,q) varying at each iteration), that one has $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0)\in\bigcap_{j=0}^{\infty}C^j(G)$, where each space $C^j(G)$ is as defined as below (6.5.49), in the proof of Lemma 6.5.8. Therefore, given the equality in (6.5.50), it follows that we have $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0)\in C^\infty(G)$; since we showed earlier that the function $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0):G\to\mathbb{C}$ is Γ-automorphic, this completes our proof of the result that $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0)\in C^\infty(\Gamma\backslash G)$. To complete the proof of the lemma we now have only to show that the function $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0):G\to\mathbb{C}$ is of K-type (ℓ,q).

By (6.5.86), the definition (6.5.82), the result (6.5.76) of Lemma 6.5.14 (applied with the K-type (λ,κ) substituted for the K-type (ℓ,q)) and the equation (6.5.29), we may infer that, for all $\mathrm{Re}(\nu)>1$, and all $\mathbf{Y}\in\mathfrak{g}$, one has:

$$\begin{aligned}\mathbf{Y}\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) &= \sum_{(\lambda,\kappa)\in I(\ell,q)} c_{\ell,q}^{\mathbf{Y}}(\lambda,\kappa;\nu,0)\,P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\lambda,\kappa}(\nu,0) = \\ &= P^{\mathfrak{a}}\Biggl(\sum_{(\lambda,\kappa)\in I(\ell,q)} c_{\ell,q}^{\mathbf{Y}}(\lambda,\kappa;\nu,0)\,\mathbf{M}_\omega\varphi_{\lambda,\kappa}(\nu,0)\Biggr) = \\ &= P^{\mathfrak{a}}\mathbf{M}_\omega\Biggl(\sum_{(\lambda,\kappa)\in I(\ell,q)} c_{\ell,q}^{\mathbf{Y}}(\lambda,\kappa;\nu,0)\,\varphi_{\lambda,\kappa}(\nu,0)\Biggr) = P^{\mathfrak{a}}\mathbf{M}_\omega\mathbf{Y}\varphi_{\ell,q}(\nu,0)\ .\end{aligned}$$

Since we have also $\mathbf{Y}P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)=\mathbf{Y}\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ (for $\mathrm{Re}(\nu)>1$ and $\mathbf{Y}\in\mathfrak{g}$), it follows from the above equations (similarly to how the results in (6.5.54) and (6.5.55) were obtained) that, when $\mathrm{Re}(\nu)>1$, one has $\mathbf{H}_2P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)=-iqP^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ and $\Omega_{\mathfrak{k}}P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)=-\frac{1}{2}(\ell^2+\ell)P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$. By this, combined with the result (6.5.76) of Lemma 7.5.12, and the definition (6.5.82), it follows that

$$\mathbf{H}_2\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)=-iq\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\qquad(\mathrm{Re}(\nu)>1);\tag{6.5.87}$$

$$\Omega_{\mathfrak{k}}\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)=-\frac{1}{2}\left(\ell^2+\ell\right)\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\qquad(\mathrm{Re}(\nu)>1).\tag{6.5.88}$$

Moreover, by iteration of the equation (6.5.86) (in the manner previously discussed) and application of the result (6.5.74) of Lemma 6.5.14, it may be shown that, when $g\in G$, $J\in\mathbb{N}$ and $\mathbf{X}_1,\ldots,\mathbf{X}_J\in\mathfrak{g}$, the function $\nu\mapsto(\mathbf{X}_J\mathbf{X}_{J-1}\cdots\mathbf{X}_1\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$ is holomorphic on the half-plane where $\mathrm{Re}(\nu)>1/2$; and so one finds, in particular, that both sides of each equation in (6.5.87) and (6.5.88) are functions of ν that are holomorphic when $\mathrm{Re}(\nu)>1/2$. The equations in (6.5.87) and (6.5.88) must therefore be valid for all $\nu\in\mathbb{C}$ such that $\mathrm{Re}(\nu)>1/2$, so we have the required proof that $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0)$ is of K-type (ℓ,q) ■

For each $\nu\in\mathbb{C}$ such that $\mathrm{Re}(\nu)>1/2$, we define now the function $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0):G\to\mathbb{C}$ by setting

$$\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)=\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)-P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\tag{6.5.89}$$

(it being a corollary of Lemma 6.5.2, the definition (6.5.82) and Lemma 6.5.14 that, when $\mathrm{Re}(\nu)>1/2$, the expression on the right-hand side of Equation (6.5.89) denotes a well-defined complex-valued function on G). In the next lemma we establish certain useful properties of the functions defined in (6.5.89): the lemma implies (amongst other things) the existence of the limit in the definition (6.5.5). Note that in (6.5.94) we use '$L^p(\Gamma\backslash G;\ell,q)$' (for $p=2/\varepsilon$) to denote the space $\{f\in L^p(\Gamma\backslash G): f \text{ is of } K\text{−type } (\ell,q)\}$.

Lemma 6.5.16. *Let $0\neq\omega\in\mathfrak{O}$ and $\mathcal{N}=\{\nu\in\mathbb{C}:\mathrm{Re}(\nu)>1/2\}$; for $\nu\in\mathcal{N}$, let $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0):G\to\mathbb{C}$ be as defined in (6.5.89); for $(\nu,g)\in\mathcal{N}\times G$, put $\Phi_\tau^\infty(\nu,g)=(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$. Then the following are valid statements:*

when $\mathrm{Re}(\nu)>1$, *one has* $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)=P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$; (6.5.90)

when $g\in G$, the function $\nu\mapsto\Phi_\tau^\infty(\nu,g)$ is holomorphic on $\mathcal{N}$; (6.5.91)

the function $(\nu,g)\mapsto\Phi_\tau^\infty(\nu,g)$ is continuous on $\mathcal{N}\times G$; (6.5.92)

when $\nu\in\mathcal{N}$, the function $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ lies in $C^\infty(\Gamma\backslash G)$ and is of K-type (ℓ,q); (6.5.93)

when $|\nu-1|<\varepsilon<1/2$, one has $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\in L^{2/\varepsilon}(\Gamma\backslash G;\ell,q)$; (6.5.94)

when $\mathrm{Re}(\nu)\geq 1$, *one has* $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\in L^\infty(\Gamma\backslash G;\ell,q)$. (6.5.95)

Moreover, when $t_1 \in (0,\infty)$, $1/2 < \sigma_1 < \sigma_2 < \infty$ *and*

$$\mathcal{R} = \mathcal{R}\left(\sigma_1,\sigma_2,t_1\right) = \left\{\nu \in \mathbb{C} \,:\, \sigma_1 \leq \mathrm{Re}(\nu) \leq \sigma_2 \text{ and } |\mathrm{Im}(\nu)| \leq t_1\right\},$$

one then has

$$\Phi_\tau^\infty\left(\nu, g_{\mathfrak{b}} g\right) \ll_{\Gamma,\omega,\ell,\mathcal{R}} \left(\rho(g)\right)^{1-\mathrm{Re}(\nu)} \qquad \textit{for } \ (\nu,g) \in \mathcal{R} \times \{g \in G : \rho(g) \geq 1/|q_0|\}, \tag{6.5.96}$$

where q_0 *denotes the 'level' of the Hecke congruence subgroup* $\Gamma \leq SL(2,\mathfrak{O})$.

Proof. Let $g_0 \in G$; and let $\nu_0 \in \mathbb{C}$ be such that $\mathrm{Re}(\nu_0) > 1$. It is then a corollary of what was found in the proofs of Lemma 6.5.1 and Lemma 6.5.2 that the Poincaré series $(P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0))(g_0)$ and $(P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0))(g_0)$ are absolutely convergent. Hence, given the linearity (implicit in (1.5.4)) of the operator $P^{\mathfrak{a}}$, it follows that the Poincaré series $(P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0))(g_0) = P^{\mathfrak{a}}\big(\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0) - (1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0)\big)(g_0)$ is absolutely convergent, and that $(P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0))(g_0) = (P^{\mathfrak{a}}\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0))(g_0) - (P^{\mathfrak{a}}(1-\tau)\mathbf{M}_\omega\varphi_{\ell,q}(\nu_0,0))(g_0)$. By this, combined with the result (6.5.76) of Lemma 6.5.14, and the definitions (6.5.82), (6.5.89), we infer what is stated in (6.5.90). Moreover, given those definitions, (6.5.82) and (6.5.89), the results in (6.5.91), (6.5.92) and (6.5.93) are an immediate corollary of the combined results of Lemma 6.5.2, Lemma 6.5.14 (the results (6.5.74), (6.5.75) in particular) and Lemma 6.5.15. The remainder of this proof is therefore devoted to the demonstration of what is asserted in (6.5.94)-(6.5.96).

Henceforth let $\mathcal{R}(a,b,c)$ denote (when $a,b,c \in \mathbb{R}$) the set $\{\nu \in \mathbb{C} : a \leq \mathrm{Re}(\nu) \leq b \text{ and } |\mathrm{Im}(\nu)| \leq c\}$. Suppose that $t_1 \in (0,\infty)$, that $1/2 < \sigma_1 < \sigma_2 < \infty$, and that $\mathcal{R} = \mathcal{R}(\sigma_1,\sigma_2,t_1)$. Then, by the definitions (6.5.89), (6.5.82) and the results (6.5.73) and (6.5.8) of Lemma 6.5.14 and Lemma 6.5.2, one has

$$\Phi_\tau^\infty\left(\nu, g_{\mathfrak{b}} g\right) \ll_{\Gamma,\omega,\ell,\mathcal{R}} \left(\rho(g)\right)^{1-\mathrm{Re}(\nu)} \qquad \text{for } \ (\nu,g) \in \mathcal{R} \times \{g \in G : \rho(g) \geq 2\}. \tag{6.5.97}$$

Observe now that $\Phi_\tau^\infty(\nu, g_{\mathfrak{b}} n[\alpha] g) = \Phi_\tau^\infty(\nu, g_{\mathfrak{b}} g)$ for $(\nu,g) \in \mathcal{N} \times G$ and all $\alpha \in \mathfrak{O}$ (this follows, given our assumption that $g_{\mathfrak{b}}^{-1}\Gamma'_{\mathfrak{b}} g_{\mathfrak{b}} = \{n[\alpha] : \alpha \in \mathfrak{O}\}$, by virtue of the result (6.5.93) proved earlier). Therefore, given the compactness of the subset $\mathcal{R} \times \{n[z]a[r]k : \mathrm{Re}(z), \mathrm{Im}(z) \in [-1/2,1/2], r \in [1/|q_0|, 2] \text{ and } k \in K\}$ of $\mathcal{N} \times G$, and since we have (see (6.5.92)) already established the continuity of the function $(\nu,g) \mapsto \Phi_\tau^\infty(\nu,g)$, it follows that the function $(\nu,g) \mapsto \Phi_\tau^\infty(\nu, g_{\mathfrak{b}} g)$ is bounded on the set $\mathcal{R} \times \{g \in G : 1/|q_0| \leq \rho(g) \leq 2\}$. Hence, by considering all the factors upon which that upper bound may depend, we find that

$$\Phi_\tau^\infty\left(\nu, g_{\mathfrak{b}} g\right) \ll_{\Gamma,\omega,\ell,\mathcal{R}} \left(\rho(g)\right)^{1-\mathrm{Re}(\nu)} \qquad \text{for } \ (\nu,g) \in \mathcal{R} \times \{g \in G : 1/|q_0| \leq \rho(g) \leq 2\}. \tag{6.5.98}$$

This last result merits some further explanation, since it would appear (at first sight) that the relevant implicit constant might have to depend on $g_{\mathfrak{a}}$, $g_{\mathfrak{b}}$ and τ. In fact the question of the dependence of this implicit constant on τ does not arise, since our choice of τ is fixed (i.e. it is clear from (6.5.1), and the discussion preceding it, that we may define our fixed choice of the function τ in purely absolute terms).

In considering whether or not the implicit constant in (6.5.98) has to depend on the scaling matrix $g_{\mathfrak{b}}$, we note firstly that, if $\mathfrak{b}' \stackrel{\Gamma}{\sim} \mathfrak{b}$, and if $g_{\mathfrak{b}}, g_{\mathfrak{b}'} \in G$ are chosen so that (1.1.16) and (1.1.20) are satisfied for $\mathfrak{c} \in \{\mathfrak{b},\mathfrak{b}'\}$, then, since $g \mapsto \Phi_\tau^\infty(\nu,g)$ is Γ-automorphic (when $\nu \in \mathcal{N}$), it may be deduced from Lemma 4.1 that when M and M' denote the global maxima attained on the set $\mathcal{R} \times \{g \in G : 1/|q_0| \leq \rho(g) \leq 2\}$ by the functions $(\nu,g) \mapsto |\Phi_\tau^\infty(\nu, g_{\mathfrak{b}} g)|$ and $(\nu,g) \mapsto |\Phi_\tau^\infty(\nu, g'_{\mathfrak{b}} g)|$ (respectively) one will have $M' = M$. Therefore the implicit constant in (6.5.98) depends on $g_{\mathfrak{b}}$ only to the extent that it depends on the Γ-equivalence class of the cusp $\mathfrak{b}$. A similar phenomenon may be observed in respect of the dependence of the same constant upon the scaling matrix $g_{\mathfrak{a}}$: for it follows by Lemma 4.1 and the equations (1.5.4), (6.3.3) and (6.3.5) that if $\mathfrak{a}' \stackrel{\Gamma}{\sim} \mathfrak{a}$, and if $g_{\mathfrak{a}}, g_{\mathfrak{a}'} \in G$ are chosen so that (1.1.16) and (1.1.20) are satisfied for $\mathfrak{c} \in \{\mathfrak{a},\mathfrak{a}'\}$, then there exists some $\beta \in \mathbb{C}$ and some $\epsilon \in \mathfrak{O}^*$ such that one has $P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) = \mathrm{e}(-\mathrm{Re}(\omega\beta))P^{\mathfrak{a}'}\tau\mathbf{M}_{\epsilon\omega}\varphi_{\ell,q}(\nu,0)$ for $\mathrm{Re}(\nu) > 1$; and this, by (6.5.90), implies that one must in fact have $\mathcal{P}_{\leftrightarrows}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) = \mathrm{e}(-\mathrm{Re}(\omega\beta))\mathcal{P}_{\leftrightarrows}^{\mathfrak{a}'}\tau\mathbf{M}_{\epsilon\omega}\varphi_{\ell,q}(\nu,0)$ for all $\nu \in \mathcal{N}$. Therefore, by using a relation of the form

$$c_0\left(\Gamma\mathfrak{a}, \Gamma\mathfrak{b}\right) \leq \max_{(\mathfrak{a}',\mathfrak{b}') \in \mathfrak{C}(\Gamma)\times\mathfrak{C}(\Gamma)} c_0\left(\Gamma\mathfrak{a}', \Gamma\mathfrak{b}'\right)$$

(in which $\mathfrak{C}(\Gamma)$ denotes any complete set of representatives of the Γ-equivalence classes of cusps, so that by Lemma 2.2 one has $|\mathfrak{C}(\Gamma)| < \infty$), we are able to obtain (6.5.98) with an implicit constant c_0^* independent of $g_{\mathfrak{a}}$, $g_{\mathfrak{b}}$, $\mathfrak{a}$, $\mathfrak{b}$ and the Γ-equivalence classes $\Gamma\mathfrak{a}$, $\Gamma\mathfrak{b}$ (except inasmuch as c_0^* may depend upon the group Γ).

By (6.5.98) and (6.5.97) we obtain the result stated in (6.5.96). In order to complete the proof of the lemma, we show next that the bound (6.5.96) implies the results stated in (6.5.94) and (6.5.95).

We embark firstly upon the deduction of (6.5.95). By reasoning similar to (but simpler than) that which precedes our statement of the bound (6.5.98), we deduce from the results in (6.5.93) that, when $\nu \in \mathcal{N}$, the function $g \mapsto \Phi_\tau^\infty(\nu, g) = (\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$ is Γ-automorphic (on G), and is bounded on each compact set $\mathcal{D}$ contained in G. Hence, since there exists (in respect of the action of Γ on the upper half-space $\mathbb{H}_3$) a fundamental domain $\mathcal{F}_*$ fitting the description given in (1.1.22)-(1.1.24), and since, for such a domain $\mathcal{F}_*$, the set $\bigcup_{(z,r)\in\mathcal{F}_*} n[z]a[r]K$ will contain a fundamental domain for the action of Γ on the group G, it follows by (1.1.24), (1.1.23) and the case $\mathcal{R} = \mathcal{R}(1, \mathrm{Re}(\nu)+1, |\mathrm{Im}(\nu)|+1)$ of the result (6.5.96) that, when $\mathrm{Re}(\nu) \geq 1$, the Γ-automorphic function $g \mapsto (\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$ is bounded on G (we use here the fact that, by (6.1.25), one has $1/|m_{\mathfrak{c}}| \geq 1/|q_0|$ for all cusps $\mathfrak{c} \in \mathbb{Q}(i)\cup\{\infty\}$). The result (6.5.95) therefore follows (given the content of the result (6.5.93) proved earlier).

We now have only to show that (6.5.94) holds. In light of the result (6.5.93) obtained earlier, it will suffice that we show that

$$\int_{\Gamma\backslash G} \left|\left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g)\right|^{2/\varepsilon} \mathrm{d}g < \infty \qquad \text{when } \ |\nu - 1| < \varepsilon < 1/2. \tag{6.5.99}$$

Let $0 < \varepsilon < 1/2$. By the case $\mathcal{R} = \mathcal{R}(1-\varepsilon, 1+\varepsilon, \epsilon)$ of the result (6.5.96), it follows that, for $|\nu - 1| < \varepsilon$ and $g \in G$ such that $\rho(g) \geq 1/|q_0|$, one has

$$\left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g_{\mathfrak{b}}g) \ll_{\Gamma,\omega,\ell,\varepsilon} (\rho(g))^{1-\mathrm{Re}(\nu)} \leq |q_0|^{2\varepsilon} (\rho(g))^{|1-\nu|} \ .$$

Hence when $|\nu - 1| < \varepsilon$ we have, in particular,

$$\left|\left(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right)(g_{\mathfrak{b}}g)\right|^{2/\varepsilon} \ll_{\Gamma,\omega,\ell,\varepsilon} (\rho(g))^{2|\nu-1|/\varepsilon} \qquad \text{for all } \ g \in G \ \text{ such that } \ \rho(g) \geq 1/|m_{\mathfrak{b}}|. \tag{6.5.100}$$

Since the hypothesis that $|\nu - 1| < \varepsilon$ implies that, in (6.5.100), the final exponent $2|\nu-1|/\varepsilon$ is strictly less than 2, it may therefore be deduced, by reasoning similar to that seen in the proof of Corollary 6.2.10, that the bound (6.5.100) is sufficient to justify what is asserted in (6.5.99) ∎

Lemma 6.5.17. *When $g \in G$ the limit on the right-hand side of Equation (6.5.5) exists. The function $P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta : G \rightarrow \mathbb{C}$ defined by Equation (6.5.5) is equal to the function $P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0)$, and lies in $C^\infty(G) \cap L^\infty(\Gamma\backslash G;\ell,q)$.*

Proof. By the definition (1.5.4), the results of Lemma 6.5.1 and Lemma 6.5.8 concerning $P^{\mathfrak{a}}|\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)|$ and $P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta|$, and the result (6.5.90) of Lemma 6.5.16, it follows that when $\mathrm{Re}(\nu) > 1$ one has

$$\begin{aligned} P^{\mathfrak{a}}\left(\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\right) &= P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) = \\ &= P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) \ . \end{aligned}$$

By this observation, and the result (6.5.91) of Lemma 6.5.16, we find that when $g \in G$ the limit on the right-hand side of Equation (6.5.5) exists, and is equal to $(P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta)(g) + b(\omega;\ell,q;\eta)(\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0))(g)$. This completes the proof of the first two assertions of the lemma; since it has, in particular, been shown that $P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta = P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0)$, the result that $P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta \in C^\infty(G) \cap L^\infty(\Gamma\backslash G;\ell,q)$ follows by virtue of Lemma 6.5.8 and the results (6.5.93) and (6.5.95) of Lemma 6.5.16 ∎

§6.6 The preliminary spectral summation formula.

Throughout this subsection we suppose that $\mathfrak{a}$, $\mathfrak{b}$, $g_{\mathfrak{a}}$, $g_{\mathfrak{b}}$, σ and the K-type (ℓ,q) are as stated at the beginning of Subsection 6.5: in particular, we suppose that $1<\sigma<2$. We assume, moreover, that ω_1 and ω_2 are non-zero Gaussian integers, and that η and θ are functions that lie in $\mathcal{T}_\sigma^\ell$ (the space defined in and below (6.4.3)). Subject to these hypotheses, we seek to establish the following result.

Proposition 6.6.1 (preliminary sum formula). *Let*

$$\phi_1 = P^{\mathfrak{a},*}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta \qquad \text{and} \qquad \phi_2 = P^{\mathfrak{b},*}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\ . \tag{6.6.1}$$

Then

$$\begin{aligned} &\sum_V \overline{C_V^{\mathfrak{a}}\left(\omega_1;\nu_V,p_V\right)}\, C_V^{\mathfrak{b}}\left(\omega_2;\nu_V,p_V\right) h_\ell\left(\nu_V,p_V\right) + \\ &\quad + \sum_{\mathfrak{c}\in\mathfrak{C}(\Gamma)} \frac{1}{4\pi i\left[\Gamma_{\mathfrak{c}}:\Gamma_{\mathfrak{c}}'\right]} \sum_{p\in\frac{1}{2}[\Gamma_{\mathfrak{c}}:\Gamma_{\mathfrak{c}}']\mathbb{Z}} \int_{(0)} \overline{B_{\mathfrak{c}}^{\mathfrak{a}}\left(\omega_1;\nu,p\right)}\, B_{\mathfrak{c}}^{\mathfrak{b}}\left(\omega_2;\nu,p\right) h_\ell(\nu,p)\,\mathrm{d}\nu = \\ &\qquad = \frac{\left[\Gamma_{\mathfrak{a}}:\Gamma_{\mathfrak{a}}'\right]\left[\Gamma_{\mathfrak{b}}:\Gamma_{\mathfrak{b}}'\right]}{4\pi^2}\left\langle \phi_1,\phi_2\right\rangle_{\Gamma\backslash G} = \\ &\qquad = \frac{1}{4\pi^3 i}\,\delta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}} \sum_{p\in\mathbb{Z}} \int_{(0)} h_\ell(\nu,p)\left(p^2-\nu^2\right)\mathrm{d}\nu + \sum_{c\in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}\left(\omega_1,\omega_2;c\right)}{|c|^2}\left(\mathbf{B}h_\ell\right)\!\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c}\right), \end{aligned} \tag{6.6.2}$$

where, for $(\nu,p)\in\mathbb{C}\times\mathbb{Z}$ such that $|\mathrm{Re}(\nu)|\le\sigma$, one has

$$h_\ell(\nu,p) = \begin{cases} \lambda_\ell^*(\nu,p)\,\overline{\theta\left(-\overline{\nu},p\right)}\,\eta(\nu,p) & \text{if } |p|\le\ell, \\ 0 & \text{otherwise,} \end{cases} \tag{6.6.3}$$

with

$$\lambda_\ell^*(\nu,p) = \Gamma(\ell+1+\nu)\Gamma(\ell+1-\nu)\,\frac{\sin^2(\pi\nu)}{(\pi\nu)^2}\,\frac{\nu^{2+2\epsilon(p)}}{\left(\nu^2-p^2\right)^2} = \frac{1}{\Gamma(\ell+1+\nu)\Gamma(\ell+1-\nu)} \prod_{\substack{0<m\le\ell \\ m\ne|p|}}\left(\nu^2-m^2\right)^2 \tag{6.6.4}$$

($\epsilon(p)$ being as in (6.4.5)), while the term $\delta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}$ and $\mathbf{B}$-transform are as defined in (1.9.2)-(1.9.6), and all other nonstandard notation has the meaning assigned to it in Subsection 1.1, Subsection 1.5, Subsection 1.7, Subsection 1.8 and (1.2.2). The sums and integrals occurring in Equation (6.6.2) are absolutely convergent.

We assume henceforth (in this subsection) that ϕ_1 and ϕ_2 are as stated in (6.6.1) (the relevant terminology having been defined in the equation (6.5.5)). By Lemma 6.5.17 and the observation (6.5.7), we have $L^2(\Gamma\backslash G)\supseteq L^\infty(\Gamma\backslash G)\ni\phi_j$ for $j=1,2$. Therefore it follows by the Cauchy-Schwarz inequality of Section 12.41 of [43] that the inner product $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$ exists.

Our proof of the above proposition occupies the remainder of this subsection. It is modelled on Bruggeman and Motohashi's original proof (in Section 9 of [5]) of the special case $\Gamma=SL(2,\mathfrak{O})$, $\mathfrak{a}=\mathfrak{b}=\infty$ of (6.6.2). In particular, we 'compute' the value of the inner product $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$ in two ways; one computation yielding the 'geometric description' of $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$ implied by the final equality in (6.6.2); the other supplying the 'spectral description' (of the same quantity) implied by the first equality of (6.6.2). We remark that these 'geometric' and 'spectral' descriptions depend on the K-type (ℓ,q) only insofar as they depend on the parameter ℓ: we shall in fact only ever need to apply the 'preliminary sum formula' (6.6.2) in respect of cases where the K-type is $(\ell,0)$.

The next lemma (derived from Lemma 6.2.4) is of key importance in enabling both the computation of the geometric description of $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$ and the computation of the corresponding spectral description. Before stating the lemma we clarify that henceforth $L^p(N\backslash G)$ will denote (when $1\le p<\infty$) the space of

those measurable functions $f : G \to \mathbb{C}$ that satisfy $f(ng) = f(g)$, for all $n \in N$, $g \in G$, and are such that $\int_{N\backslash G} |f(g)|^p \, \mathrm{d}\dot{g} < \infty$ (where the measure $\mathrm{d}\dot{g}$ is that which occurs in (6.2.9)).

Lemma 6.6.2. *Let $0 \neq \omega \in \mathfrak{O}$, let $\alpha, \beta \in (0,\infty)$, and let $\phi \in C^0(G) \cap L^{1+\alpha}(\Gamma\backslash G)$. Suppose moreover that one has $|F_\omega^{\mathfrak{a}}\phi| \in L^{1+\beta}(N\backslash G)$. Then*

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \left\langle P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta \,, \phi \right\rangle_{\Gamma\backslash G} = \left\langle \widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta \,, F_\omega^{\mathfrak{a}}\phi \right\rangle_{N\backslash G} . \tag{6.6.5}$$

Proof. It will suffice to show that one has both

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \left\langle P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta \,, \phi \right\rangle_{\Gamma\backslash G} = \left\langle \widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta \,, F_\omega^{\mathfrak{a}}\phi \right\rangle_{N\backslash G} \tag{6.6.6}$$

and

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \left\langle \mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0) \,, \phi \right\rangle_{\Gamma\backslash G} = \left\langle \tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0) \,, F_\omega^{\mathfrak{a}}\phi \right\rangle_{N\backslash G} . \tag{6.6.7}$$

Indeed, the equalities asserted in (6.6.6) and (6.6.7) imply the equality

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \left\langle P^{\mathfrak{a}}\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0) \,, \phi \right\rangle_{\Gamma\backslash G} = \left\langle \widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta + b(\omega;\ell,q;\eta)\tau\mathbf{M}_\omega\varphi_{\ell,q}(1,0) \,, F_\omega^{\mathfrak{a}}\phi \right\rangle_{N\backslash G} ,$$

which, by Lemma 6.5.17 and the definition (6.5.2), is equivalent to Equation (6.6.5).

The equality in (6.6.6) is just the case $f_\omega = \widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta$ of the result (6.2.8) of Lemma 6.2.4. It therefore suffices for proof of (6.6.6) that we verify that the hypotheses of Lemma 6.2.4 are satisfied when $f_\omega = \widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta$ and ϕ is as we suppose (in this proof). This (since the relation $\phi \in C^0(\Gamma\backslash G)$ is implicit in our current hypotheses) merely entails our showing that $(P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta|) \cdot \phi \in L^1(\Gamma\backslash G)$, that $\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta \in C^0(N\backslash G, \omega)$, and that, for some $\sigma_0 > 1$ and some $R_0 > 0$, one has

$$\left(\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta\right)(g) \ll_{\omega,\eta,\sigma_0,R_0} \left(\rho(g)\right)^{1+\sigma_0} \qquad \text{for all } g \in G \text{ such that } \rho(g) \leq R_0. \tag{6.6.8}$$

Since we assume (throughout this subsection) that $\eta \in \mathcal{T}^{\ell}_{\sigma}$, and that $\sigma \in (1,2)$, it follows by the observation recorded in (6.5.22) that, if one puts $R_0 = 1$ and $\sigma_0 = \sigma > 1$, then one does have $R_0 > 0$, $\sigma_0 > 1$ and the desired growth estimate (6.6.8). By (6.5.4), we have also that $\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta \in C^0(N\backslash G, \omega)$. Therefore the equality (6.6.6) follows if $(P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta|) \cdot \phi \in L^1(\Gamma\backslash G)$. By Hölder's inequality (as formulated in Section 12.42 of [43]), we find that the last condition on $(P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta|) \cdot \phi$ is satisfied: for we have $L^{1+\alpha}(\Gamma\backslash G) \ni \phi$, by hypothesis, and it follows by Lemma 6.5.8 and the observation (6.5.7) that $L^{1+1/\alpha}(\Gamma\backslash G) \supseteq L^{\infty}(\Gamma\backslash G) \ni P^{\mathfrak{a}}|\widetilde{\mathbf{L}}^{\omega,*}_{\ell,q}\eta|$. This completes the proof of (6.6.6).

In what follows we take $\Phi_1(\nu)$ and $\Phi_2(\nu)$ to denote the inner products $\left\langle \mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) \,, \phi \right\rangle_{\Gamma\backslash G}$ and $\left\langle \tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) \,, F_\omega^{\mathfrak{a}}\phi \right\rangle_{N\backslash G}$, respectively (so that each of $\Phi_1(\nu)$, $\Phi_2(\nu)$ is defined only for those $\nu \in \mathbb{C}$ such that the relevant inner product exists). We show next that

$$[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] \, \Phi_1(\nu) = \Phi_2(\nu) \qquad \text{for all } \nu \in \mathbb{C} \text{ such that } \mathrm{Re}(\nu) > 1. \tag{6.6.9}$$

By the result (6.5.90) of Lemma 6.5.16, we have $\Phi_1(\nu) = \langle P^{\mathfrak{a}}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) \,, \phi\rangle_{\Gamma\backslash G}$ when $\mathrm{Re}(\nu) > 1$. This suggests that we might prove (6.6.9) by another application of Lemma 6.2.4, similar to that (above) by which the equality (6.6.6) was obtained (though with $f_\omega = \tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)$ on this occasion). As previously, we note that our hypotheses imply that $\phi \in C^0(\Gamma\backslash G)$. Furthermore, by (6.5.1), (6.5.3) and the result (6.3.7) of Lemma 6.3.1, we have $(\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g) \ll_{\omega,\ell,\nu} (\rho(g))^{1+\mathrm{Re}(\nu)}$ for $\nu \in \mathbb{C}$, $g \in G$, and, given our choice of $\tau \in C^{\infty}(G)$ (as in (6.5.1)) and the observation preceding (6.3.4), we have also $\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0) \in C^{\infty}(N\backslash G, \omega)$ for all $\nu \in \mathbb{C}$. Therefore, if it is the case that $(P^{\mathfrak{a}}|\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)|) \cdot \phi \in L^1(\Gamma\backslash G)$ when $\mathrm{Re}(\nu) > 1$, then Lemma 6.2.4 applies, giving (6.6.9). By Hölder's inequality and the hypothesis that $\phi \in L^{1+\alpha}(\Gamma\backslash G)$, we find that $(P^{\mathfrak{a}}|\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)|) \cdot \phi \in L^1(\Gamma\backslash G)$ when $P^{\mathfrak{a}}|\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)| \in L^{1+1/\alpha}(\Gamma\backslash G)$; since Lemma 6.5.1 and the

observation (6.5.7) imply that one has $L^{1+1/\alpha}(\Gamma\backslash G) \supseteq L^\infty(\Gamma\backslash G) \ni P^{\mathfrak{a}}|\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)|$ when $\mathrm{Re}(\nu) > 1$, this completes the proof of (6.6.9).

We complete the proof of the lemma by showing that

$$\Phi_j(1) = \lim_{\nu\to 1+} \Phi_j(\nu) \qquad \text{for } j = 1,2. \tag{6.6.10}$$

This suffices for completion of the proof, since the combination of (6.6.9) and (6.6.10) implies the equation $[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]\,\Phi_1(1) = \Phi_2(1)$, which is (6.6.7). The approach that we take to our proof of (6.6.10) is to establish the stronger result that each of the functions $\Phi_1(\nu), \Phi_2(\nu)$ is continuous in some neighbourhood of the point $\nu = 1$. Before we proceed, note that by the relations in (6.5.7) there is no loss of generality in assuming henceforth that $0 < \alpha < 1/3$ (this helps to simplify some of the calculations below).

We prove first the case $j = 1$ of (6.6.10). In doing so we assume (as we may) that the sets $\mathfrak{C}(\Gamma)$ and $\mathcal{D}$, and the family $(\mathcal{E}_{\mathfrak{c}})_{\mathfrak{c}\in\mathfrak{C}(\Gamma)}$, are each as described in the final paragraph of Subsection 1.1; consequently $\mathfrak{C}(\Gamma)$ is (by Lemma 2.2) a finite set of cusps, and the set $\mathcal{F}_*$ given by Equation (1.1.24) is a fundamental domain for the action of Γ on $\mathbb{H}_3$. By reasoning similar to that already seen in the first paragraph of the proof of Corollary 6.2.10, we find that the case $j = 1$ of (6.6.10) certainly follows if, when $\mathcal{X} \in \{\mathcal{D}\} \cup \{\mathcal{E}_{\mathfrak{c}} : \mathfrak{c} \in \mathfrak{C}(\Gamma)\}$, the complex function

$$\nu \mapsto \frac{1}{2}\int_{\mathcal{X}}\int_K \big(\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\big)(n[z]a[r]k)\,\overline{\phi(n[z]a[r]k)}\,\mathrm{d}k\, r^{-3}\mathrm{d}_+z\mathrm{d}r \tag{6.6.11}$$

is continuous in some neighbourhood of the point $\nu = 1$.

Note that, since $\phi \in C^0(\Gamma\backslash G)$, the result (6.5.92) of Lemma 6.5.16 implies that the integrand in (6.6.11) is continuous, as a function of (ν, g), on the set $\{\nu \in \mathbb{C} : \mathrm{Re}(\nu) > 1/2\} \times G$. Therefore, by a proof similar in principle to that in Section 1.52 of [43] (on 'The continuity theorem'), it follows that if the set $\tilde{\mathcal{X}} = \{n[z]a[r]k : k \in K,\ (z,r) \in \mathcal{X}\}$ is a compact measurable subset of $G = SL(2,\mathbb{C})$ then the function (6.6.11) is continuous in the open half-plane where $\mathrm{Re}(\nu) > 1/2$. Hence, in the particular case where $\mathcal{X} = \mathcal{D}$ (a compact hyperbolic polyhedron, with finitely many faces), it follows by virtue of the compactness of $K = SU(2)$ that the function (6.6.11) is certainly continuous in the neighbourhood $\{\nu \in \mathbb{C} : |\nu - 1| < 1/2\}$ of the point $\nu = 1$.

Suppose that, for some $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, we have $\mathcal{X} = \mathcal{E}_{\mathfrak{c}}$ (so that, in (6.6.11), '$\mathcal{X}$' denotes some non-compact 'cusp-sector' within the hyperbolic upper half-space $\mathbb{H}_3$). Then, by (1.1.23), the integral in (6.6.11) is

$$\int_{\tilde{\mathcal{E}}_{\mathfrak{c}}} \big(\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\big)(g)\,\overline{\phi(g)}\,\mathrm{d}g = \int\limits_{1/|m_{\mathfrak{c}}|}^{\infty}\int_{\mathcal{R}_{\mathfrak{c}}}\int_K \big(\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\big)(g_{\mathfrak{c}}n[z]a[r]k)\,\overline{\phi\left(g_{\mathfrak{c}}n[z]a[r]k\right)}\,\mathrm{d}k\,\mathrm{d}_+z\,\frac{\mathrm{d}r}{r^3}\ ,$$

where $\tilde{\mathcal{E}}_{\mathfrak{c}} = \{n[z]a[r]k : k \in K,\ (z,r) \in \mathcal{E}_{\mathfrak{c}}\}$ and $\mathcal{R}_{\mathfrak{c}}$ is the bounded rectangular region of the complex plane defined in (1.1.22). Since G is a topological group it follows, by (1.1.3) and what has been noted in the preceding paragraph, that the latter integrand (above) is continuous, as a function of (ν, z, r, k), on the set $\{\nu \in \mathbb{C} : \mathrm{Re}(\nu) > 1/2\} \times \mathbb{C} \times (0,\infty) \times K$. Therefore, in order to justify a similar conclusion to that reached at the end of the previous paragraph (in respect of the case $\mathcal{X} = \mathcal{D}$), it is enough that we establish a certain uniformity of convergence of the above integral over the set $(1/|m_{\mathfrak{c}}|,\infty) \times \mathcal{R}_{\mathfrak{c}} \times K$: indeed, since each set in the family $([1/|m_{\mathfrak{c}}|, r] \times \mathcal{R}_{\mathfrak{c}} \times K)_{r\in\mathbb{N}}$ is compact, it suffices that we find some $\delta > 0$ such that

$$\lim_{r_0\to+\infty}\ \sup_{|\nu-1|\le\delta}\left|\ \int\limits_{r_0}^{\infty}\int_{\mathcal{R}_{\mathfrak{c}}}\int_K \big(\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\big)(g_{\mathfrak{c}}n[z]a[r]k)\,\overline{\phi\left(g_{\mathfrak{c}}n[z]a[r]k\right)}\,\mathrm{d}k\,\mathrm{d}_+z\,\frac{\mathrm{d}r}{r^3}\ \right| = 0\ . \tag{6.6.12}$$

A $\delta > 0$ such that (6.6.12) holds may be determined by applying the results (6.5.94) and (6.5.96) of Lemma 6.5.16: for, by the Hölder inequality of Section 12.42 of [43], the hypothesis that $\phi \in L^{1+\alpha}(\Gamma\backslash G)$, the case $\epsilon = 2/(1+1/\alpha) \in (0,1/2)$ of (6.5.94), and the case $t_1 = 2/(1+1/\alpha)$, $\sigma_1 = 1 - t_1$, $\sigma_2 = 1 + t_1$

of (6.5.96), it follows that, when $|\nu - 1| < 2/(1+1/\alpha)$, one has $(\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)) \cdot \overline{\phi} \in L^1(\Gamma\backslash G)$ and, for $r_0 \geq 1$,

$$\Bigl| \int\limits_{r_0}^{\infty} \int_{\mathcal{R}_{\mathfrak{c}}} \int_K \bigl(\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\bigr)(g_{\mathfrak{c}}n[z]a[r]k)\, \overline{\phi\left(g_{\mathfrak{c}}n[z]a[r]k\right)} \mathrm{d}k\, \mathrm{d}_+z\, \frac{\mathrm{d}r}{r^3} \Bigr|^{1+1/\alpha} \leq$$

$$\leq \left(2\int_{\Gamma\backslash G} |\phi(g)|^{1+\alpha}\mathrm{d}g\right)^{1/\alpha} \int\limits_{r_0}^{\infty} \int_{\mathcal{R}_{\mathfrak{c}}} \int_K \bigl|\bigl(\mathcal{P}^{\mathfrak{a}}_{\leftarrow}\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\bigr)(g_{\mathfrak{c}}n[z]a[r]k)\bigr|^{1+1/\alpha}\, \mathrm{d}k\, \mathrm{d}_+z\, \frac{\mathrm{d}r}{r^3} =$$

$$= O_{\alpha,\phi}(1) \cdot \int\limits_{r_0}^{\infty} \int_{\mathcal{R}_{\mathfrak{c}}} \int_K O_{\Gamma,\omega,\ell,\alpha}\left(r^{(1-\mathrm{Re}(\nu))(1+1/\alpha)-3}\right) \mathrm{d}k\, \mathrm{d}_+z\, \mathrm{d}r \ll_{\Gamma,\omega,\ell,\alpha,\phi} \mu^{-1} r_0^{-\mu} ,$$

where we have $\mu = 2-(1+1/\alpha)(1-\mathrm{Re}(\nu)) \geq 2-(1+1/\alpha)|1-\nu| > 0$. Hence we find that (6.6.12) holds for any $\delta \in (0, 2/(1+1/\alpha))$, and so may deduce that, when $\mathcal{X} \in \{\mathcal{E}_{\mathfrak{c}} : \mathfrak{c} \in \mathfrak{C}(\Gamma)\}$, the function (6.6.11) is certainly continuous on the neighbourhood $\{\nu \in \mathbb{C} : |\nu - 1| \leq 1/(1+1/\alpha)\}$ of the point $\nu = 1$. Given the similar result obtained earlier in respect of the case $\mathcal{X} = \mathcal{D}$, our proof of the case $j = 1$ of (6.6.10) is complete.

We now have only to prove the case $j = 2$ of (6.6.10): the lemma will then follow. Given the definition (6.2.9) of the inner product $\langle f, F\rangle_{N\backslash G}$, it follows by (6.5.3) and (6.5.1) that we have

$$\Phi_2(\nu) = \bigl\langle \tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\, , F^{\mathfrak{a}}_\omega\phi\bigr\rangle_{N\backslash G} = \int\limits_0^2 \int_K \bigl(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\bigr)(a[r]k)\, \overline{\left(F^{\mathfrak{a}}_\omega\phi\right)(a[r]k)}\, \mathrm{d}k\, \frac{\tau(a[r])\, \mathrm{d}r}{r^3} , \tag{6.6.13}$$

for all $\nu \in \mathbb{C}$ such that the integral to the right of the second equality sign in (6.6.13) exists. As noted in the final paragraph of the proof of Lemma 6.5.2, it is a corollary of Lemma 6.1 of [5] that the function $(\nu, g) \mapsto (\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0))(g)$ is continuous on $\mathbb{C} \times G$. We have also $\tau \in C^\infty(G)$ (by choice), and, since our hypothesis that $\phi \in C^0(G) \cap L^{1+\alpha}(\Gamma\backslash G)$ implies that $\phi \in C^0(\Gamma\backslash G)$, it moreover follows from the definition (1.4.2) that $F^{\mathfrak{a}}_\omega\phi$ lies in the space $C^0(N\backslash G, \omega)$. Therefore the integrand which appears in (6.6.13) is continuous, as a function of (ν, r, k), on the set $\mathbb{C} \times (0, 2] \times K$. Consequently we may now complete the proof of the case $j = 2$ of (6.6.10) by finding some $\delta_2 > 0$ such that

$$\lim_{r_1\to 0+} \sup_{|\nu-1|\leq\delta_2} \Bigl| \int\limits_0^{r_1} \int_K \bigl(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\bigr)(a[r]k)\, \overline{\left(F^{\mathfrak{a}}_\omega\phi\right)(a[r]k)}\, \mathrm{d}k\, \frac{\tau(a[r])\, \mathrm{d}r}{r^3} \Bigr| = 0 . \tag{6.6.14}$$

By Hölder's inequality, the hypothesis that $|F^{\mathfrak{a}}_\omega\phi| \in L^{1+\beta}(N\backslash G)$, the equations in (6.5.1), and the case $r_1 = 1$, $\sigma_0 = 2$ of the estimate (6.3.9) of Lemma 6.3.1, it follows that, when $|\nu - 1| < 2/(\beta+1)$, one has $(\tau\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)) \cdot \overline{(F^{\mathfrak{a}}_\omega\phi)} \in L^1(N\backslash G)$ and, for $0 < r_1 \leq 1$,

$$\Bigl| \int\limits_0^{r_1} \int_K \bigl(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\bigr)(a[r]k)\, \overline{\left(F^{\mathfrak{a}}_\omega\phi\right)(a[r]k)}\, \mathrm{d}k\, \frac{\tau(a[r])\, \mathrm{d}r}{r^3} \Bigr|^{1+1/\beta} \leq$$

$$\leq \left(\int_{N\backslash G} \bigl|\left(F^{\mathfrak{a}}_\omega\phi\right)(g)\bigr|^{1+\beta}\, \mathrm{d}\dot{g}\right)^{1/\beta} \int\limits_0^{r_1} \int_K \bigl|\bigl(\mathbf{M}_\omega\varphi_{\ell,q}(\nu,0)\bigr)(a[r]k)\bigr|^{1+1/\beta}\, \mathrm{d}k\, \frac{\mathrm{d}r}{r^3} =$$

$$= O_{\mathfrak{a},\omega,\phi,\beta}(1) \cdot \int\limits_0^{r_1} \int_K O_{\ell,\omega}\left(r^{(1+\mathrm{Re}(\nu))(1+1/\beta)-3}\right) \mathrm{d}k\, \mathrm{d}r \ll_{\mathfrak{a},\ell,\omega,\phi,\beta} \lambda^{-1} r_1^{\lambda} ,$$

where $\lambda = (1+1/\beta)(1+\mathrm{Re}(\nu)) - 2 > (1+1/\beta)(2-2/(\beta+1)) - 2 = 0$. This implies that (6.6.14) holds for any $\delta_2 \in (0, 2/(\beta+1))$. We are therefore able to conclude that the function $\Phi_2(\nu)$ is certainly defined and

continuous on the neighbourhood $\{\nu \in \mathbb{C} : |\nu - 1| \leq 1/(\beta + 1)\}$ of the point $\nu = 1$; since the case $j = 2$ of (6.6.10) follows, this completes our proof of the lemma ■

Remark 6.6.3. Each of the two functions $\Phi_1(\nu)$, $\Phi_2(\nu)$ considered in the above proof is in fact holomorphic at all points ν satisfying a condition of the form $\mathrm{Re}(\nu) > a_j$ (where, in each case, a_j is less than 1). This may (for example) be shown by adapting the proofs of the propositions in Section 2.83 and Section 2.84 of [43].

In applying Lemma 6.6.2 to obtain the geometric description of $\langle \phi_1, \phi_2 \rangle_{\Gamma \backslash G}$ we require the assistance of the following six supplementary lemmas.

Lemma 6.6.4. *Let $\delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2} \in \mathbb{C}$ be given by the equation (1.9.2). Then*

$$F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta = \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2} \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \eta + \frac{\pi^2}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1, \omega_2; c)}{|c|^2} \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \kappa(\omega_1, \omega_2; c)\, \eta\, , \tag{6.6.15}$$

where, for $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$, the linear operator $\kappa(\omega_1, \omega_2; c)$ from $\mathcal{T}^{\ell}_{\sigma}$ into $\mathcal{T}^{\ell}_{\sigma}$ is defined as in Lemma 6.4.3.

Proof. By Lemma 6.5.17 and the definition (6.5.89),

$$P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta = P^{\mathfrak{a}} \widetilde{\mathbf{L}}^{\omega_1,*}_{\ell,q} \eta + b(\omega_1; \ell, q; \eta) \underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0) - b(\omega_1; \ell, q; \eta) P^{\mathfrak{a}} (1 - \tau) \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0)\, .$$

Since it is moreover shown by Lemma 6.5.2, Lemma 6.5.8 and Lemma 6.5.15 that each of the functions $P^{\mathfrak{a}}(1-\tau)\mathbf{M}_{\omega_1}\varphi_{\ell,q}(1,0)$, $\underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0)$ and $P^{\mathfrak{a}} \widetilde{\mathbf{L}}^{\omega_1,*}_{\ell,q} \eta$ is both continuous and Γ-automorphic on G, we may therefore deduce that

$$F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta = F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a}} \widetilde{\mathbf{L}}^{\omega_1,*}_{\ell,q} \eta - b(\eta) F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a}} (1 - \tau) \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0) + b(\eta) F^{\mathfrak{b}}_{\omega_2} \underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0)\, , \tag{6.6.16}$$

where $b(\eta) = b(\omega_1; \ell, q; \eta)$.

Let $g \in G$. By (1.4.2) and (6.5.82),

$$\left(F^{\mathfrak{b}}_{\omega_2} \underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0) \right)(g) = \int_{B^+ \backslash N} (\psi_{\omega_2}(n))^{-1} \left(\sum_{\omega' \in \mathfrak{O}} \phi_{\omega'}(1, ng) \right) \mathrm{d}n\, , \tag{6.6.17}$$

where the terms of the sum over $\omega' \in \mathfrak{O}$ are given by the case $\omega = \omega_1$ of Equation (6.5.72). For each $\omega' \in \mathfrak{O}$, the mapping $n \mapsto (\psi_{\omega_2}(n))^{-1} \phi_{\omega'}(1, ng)$ is a continuous function on N; since one has both $|\psi_{\omega_2}(n)| = 1$ and $\rho(ng) = \rho(g)$ for $n \in N$ and $g \in G$, it therefore follows by the uniformity of convergence established in Lemma 6.5.14 that one may integrate term by term in (6.6.17), so as to obtain the result

$$\left(F^{\mathfrak{b}}_{\omega_2} \underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0) \right)(g) = \sum_{\omega' \in \mathfrak{O}} \int_{B^+ \backslash N} (\psi_{\omega_2}(n))^{-1} \phi_{\omega'}(1, ng)\, \mathrm{d}n\, , \tag{6.6.18}$$

where, as follows by (6.5.72) (with ω_1 substituted for ω), the equalities (1.5.16) and (1.4.7)-(1.4.9), the identities $n[z]n[w] = n[w]n[z]$ and $h[u]n[z] = n[u^2 z]h[u]$, the equations in (6.3.3) and (1.4.3), and the definition (1.5.6) of $\delta_{\alpha,\beta}$, one has:

$$\begin{aligned} \phi_{\omega'}(1, ng) = \ & \frac{1}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{\substack{\gamma \in \Gamma'_{\mathfrak{a}} \backslash \Gamma \ :\ \gamma\mathfrak{b} = \mathfrak{a} \\ g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} \in h[u(\gamma)] N}} \delta_{\omega_1 u(\gamma), \omega'/u(\gamma)} \left(\mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0) \right) \left(g_{\mathfrak{a}}^{-1} \gamma g_{\mathfrak{b}} g \right) \psi_{\omega'}(n) + \\ & + \zeta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega'}(1) \left(\mathbf{J}_{\omega'} \varphi_{\ell,q}(1,0) \right)(g)\, \psi_{\omega'}(n)\, . \end{aligned} \tag{6.6.19}$$

Since the family $(\psi_{\omega})_{\omega \in \mathfrak{O}}$ is an orthonormal system on $B^+ \backslash N$, it follows from (6.6.18) and (6.6.19) that

$$\left(F^{\mathfrak{b}}_{\omega_2} \underset{\hookleftarrow}{\mathcal{P}}^{\mathfrak{a}} \mathbf{M}_{\omega_1} \varphi_{\ell,q}(1,0) \right)(g) = \phi_{\omega_2}(1, n[0]g)\, . \tag{6.6.20}$$

By the observations (6.5.9) and (6.5.10) noted within the proof of Lemma 6.5.2, and by the first part of (6.5.4), and the case $\theta = \eta$ of (6.5.22), it follows (since we assume $\sigma \in (1,2)$) that Lemma 6.2.5 implies, for $\mathfrak{a}' = \mathfrak{b}$, $\omega' = \omega_2$, $\omega = \omega_1$ and $f_\omega = f_{\omega_1} \in \{(1-\tau)\mathbf{M}_{\omega_1}\varphi_{\ell,q}(1,0), \widetilde{\mathbf{L}}_{\ell,q}^{\omega_1,*}\eta\}$, the applicability of the the formula for $(F_{\omega'}^{\mathfrak{a}'}P^{\mathfrak{a}}f_\omega)(g)$ stated in (1.5.5)-(1.5.10). Hence, given the definition (6.5.2), we find that

$$\begin{aligned}[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}] &\left(\left(F_{\omega_2}^{\mathfrak{b}}P^{\mathfrak{a}}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1,*}\eta\right)(g) - b\left(\omega_1;\ell,q;\eta\right)\left(F_{\omega_2}^{\mathfrak{b}}P^{\mathfrak{a}}(1-\tau)\mathbf{M}_{\omega_1}\varphi_{\ell,q}(1,0)\right)(g)\right) = \\ &= \sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma \,:\, \gamma\mathfrak{b}=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}\in h[u(\gamma)]N}} \delta_{\omega_1 u(\gamma),\omega_2/u(\gamma)}\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1,\dagger}\eta\right)(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}g) + \sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c)\left(\mathbf{J}_{\omega_2}\mathbf{h}_{1/c}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1,\dagger}\eta\right)(g)\;, \end{aligned} \tag{6.6.21}$$

where the function $\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1,\dagger}\eta : G \to \mathbb{C}$ has the definition indicated by Equation (6.5.20).

Regarding now what was observed in (6.6.19) and (6.6.20), we note that, by what was found below (6.5.68) concerning the convergence of the sum in (6.5.61), and by the result (6.3.11) of Lemma 6.3.2, the equation (6.3.5) and the linearity of the operator $\mathbf{J}_{\omega_2}$, it follows that

$$\begin{aligned}[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]\,\zeta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}(1)\,\mathbf{J}_{\omega_2}\varphi_{\ell,q}(1,0) &= \sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c)}{|c|^4}\,\mathcal{J}_{1,0}^{*}\Big(2\pi\sqrt{c^{-2}\omega_1\omega_2}\Big)\,\mathbf{J}_{\omega_2}\varphi_{\ell,q}(1,0) = \\ &= \sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c)\,\mathbf{J}_{\omega_2}\mathbf{h}_{1/c}\mathbf{M}_{\omega_1}\varphi_{\ell,q}(1,0)\;. \end{aligned}$$

Therefore, given that the definition (6.5.20) implies the identity $\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1,\dagger}\eta + b(\omega_1;\ell,q;\eta)\mathbf{M}_{\omega_1}\varphi_{\ell,q}(1,0) = \widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta$, it follows by (6.6.16), (6.6.19), (6.6.20) and (6.6.21) that

$$\begin{aligned}[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]\left(F_{\omega_2}^{\mathfrak{b}}P^{\mathfrak{a},*}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta\right)(g) = &\sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma \,:\, \gamma\mathfrak{b}=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}\in h[u(\gamma)]N}} \delta_{\omega_1 u(\gamma),\omega_2/u(\gamma)}\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta\right)(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}g) + \\ &+ \sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c)\left(\mathbf{J}_{\omega_2}\mathbf{h}_{1/c}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta\right)(g)\;, \end{aligned} \tag{6.6.22}$$

where $\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta : G \to \mathbb{C}$ is given by the case $\omega = \omega_1$ of (6.4.4)-(6.4.5).

By Lemma 2.1, the conditions imposed in (6.6.22) on the variable of summation γ imply that one has there $u^2(\gamma) = (u(\gamma))^2 \in \mathfrak{O}^*$. Consequently, given that the relevant summand is non-zero only when $\omega_1 u(\gamma) = \omega_2/u(\gamma)$, the sum over γ in (6.6.22) is effectively empty unless one has $\omega_1 \sim \omega_2$; furthermore, if $\omega_1 \sim \omega_2$, then that summation is effectively restricted to $\gamma \in \Gamma'_{\mathfrak{a}}\backslash\Gamma$ such that $g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}} = h[u]n$ for some (necessarily unique) pair $(u,n) = (u(\gamma), n[z(\gamma)]) \in \mathbb{C}^* \times N$ such that the number $\epsilon = \epsilon(\gamma) = u^2 \in \mathbb{C}^*$ satisfies $\epsilon = \omega_2/\omega_1 \in \mathfrak{O}^*$. Since it moreover follows by (6.4.4), the case $n = n[0]$ of (1.8.2), and the identity $\mathbf{h}_u\mathbf{J}_\omega\mathbf{h}_u = |u|^4\mathbf{J}_{u^2\omega}$ that $\mathbf{h}_{\pm\sqrt{\epsilon}}\widetilde{\mathbf{L}}_{\ell,q}^{\omega}\eta = \widetilde{\mathbf{L}}_{\ell,q}^{\epsilon\omega}\eta$ for $0 \neq \omega \in \mathbb{C}$ and any $\epsilon \in \mathbb{C}$ of unit modulus, and since one has $\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\eta \in C^\infty(N\backslash G, \omega_2)$ (directly by (6.4.4), or by (6.4.7)), we therefore find that

$$\begin{aligned}\sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma \,:\, \gamma\mathfrak{b}=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}\in h[u(\gamma)]N}} \delta_{\omega_1 u(\gamma),\omega_2/u(\gamma)}\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta\right)(g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}g) &= \sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma \,:\, \gamma\mathfrak{b}=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}= h[u(\gamma)]n[z(\gamma)]}} \delta_{\omega_1 u(\gamma),\omega_2/u(\gamma)}\left(\mathbf{h}_{u(\gamma)}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta\right)(n[z(\gamma)]\,g) = \\ &= \sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma \,:\, \gamma\mathfrak{b}=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}= h[u(\gamma)]n[z(\gamma)]}} \delta_{\omega_1 u(\gamma),\omega_2/u(\gamma)}\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\eta\right)(n[z(\gamma)]\,g) = \\ &= \sum_{\substack{\gamma\in\Gamma'_{\mathfrak{a}}\backslash\Gamma \,:\, \gamma\mathfrak{b}=\mathfrak{a} \\ g_{\mathfrak{a}}^{-1}\gamma g_{\mathfrak{b}}= h[u(\gamma)]n[z(\gamma)]}} \delta_{\omega_1 u(\gamma),\omega_2/u(\gamma)}\,\psi_{\omega_2}(n[z(\gamma)])\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\eta\right)(g) = \\ &= \delta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}\cdot\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\eta\right)(g)\;. \end{aligned}$$

By this, the equation (6.6.22), and the result (6.4.13) of Lemma 6.4.3, we obtain the result in (6.6.15) ■

Lemma 6.6.5. *Let $0 \neq \omega \in \mathbb{C}$. Then, for each $\beta \in (0,\infty)$, one has $L^{1+\beta}(N\backslash G) \ni |\widetilde{\mathbf{L}}^{\omega}_{\ell,q}\eta|$.*

Proof. In view of the definition (implicit in (6.2.9)) of the measure $\mathrm{d}\dot{g}$ on $N\backslash G$, this lemma is a straightforward corollary of the results (6.4.7) and (6.4.8) of Theorem 6.4.1 ■

Lemma 6.6.6 (Bruggeman and Motohashi). *Let $c \in \mathbb{C}^*$, and let $u = 2\pi\sqrt{\omega_1\omega_2}/c$. Then, for $g \in G$ and $0 < \alpha \leq \sigma$, one has*

$$\begin{aligned}\big(\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q}\kappa(\omega_1,\omega_2;c)\,\eta\big)(g) &= \\ &= \frac{(-1)}{\pi^3 i}\sum_{|p|\leq\ell}\frac{(-i\omega_2/|\omega_2|)^p}{\left\|\Phi^{\ell}_{p,q}\right\|_K}\int_{(\alpha)}\mathcal{J}_{\nu,p}(u)\,\eta(\nu,p)\,(\pi|\omega_2|)^{-\nu}\,\Gamma(\ell+1+\nu)\,\big(\mathbf{J}_{\omega_2}\varphi_{\ell,q}(\nu,p)\big)(g)\,\nu^{\epsilon(p)}\mathrm{d}\nu\,+ \\ &\quad+\frac{\ell!}{\pi^2}\sum_{0<|p|\leq\ell}\frac{(-i\omega_2/|\omega_2|)^p}{\left\|\Phi^{\ell}_{p,q}\right\|_K}\mathcal{J}_{0,p}(u)\,\eta(0,p)\,\big(\mathbf{J}_{\omega_2}\varphi_{\ell,q}(0,p)\big)(g)\end{aligned}\qquad(6.6.23)$$

where $\mathcal{J}_{\nu,p}:\mathbb{C}^*\to\mathbb{C}$ and $\epsilon:\mathbb{Z}\to\{-1,1\}$ are given by (1.9.5)-(1.9.6) and (6.4.5).

Proof. This identity is implied by the equations (7.22) and (7.24) of [5] ■

Lemma 6.6.7. *Let $p \in \mathbb{Z}$, and let $u \in \mathbb{C}^*$. Then*

$$0 \leq (-1)^p\mathcal{J}_{0,p}(u) \leq (|p|!)^{-2}|u/2|^{2|p|}\exp\left(|u|^2/2\right)\ . \qquad (6.6.24)$$

Suppose moreover that $\delta \in (0,2]$, and that $\nu \in \mathbb{C}$ is such that $\min\{|m-\nu| : m \in \mathbb{Z}\} \geq \delta$. Then one has also

$$|\mathcal{J}_{\nu,p}(u)| \leq \frac{4\delta^{-2}|u/2|^{2\mathrm{Re}(\nu)}\exp\left(|u|^2/2\right)}{|\Gamma(\nu-p+1)\Gamma(\nu+p+1)|}\ . \qquad (6.6.25)$$

Proof. By (1.9.5) and (1.9.9), one has $(-1)^p\mathcal{J}_{0,p}(u) = |u/2|^{2|p|}J^*_{|p|}(u)J^*_{|p|}(\overline{u})$. It is moreover implied by the power series representation (1.9.6) of $J^*_{\xi}(z)$ that $J^*_{|p|}(u)J^*_{|p|}(\overline{u}) = |J^*_{|p|}(u)|^2 \leq ((|p|!)^{-1}\exp(|u/2|^2))^2$, and so the result (6.6.24) follows. By (1.9.6) (again), we have also

$$\left|J^*_{\xi}(z)\right| \leq \sum_{m=0}^{\infty}\frac{|z/2|^{2m}}{(m!)(1/2)\delta|\Gamma(\xi+1)|} = \frac{2\delta^{-1}\exp\left(|z/2|^2\right)}{|\Gamma(\xi+1)|} \qquad (z \in \mathbb{C}^*)$$

whenever $\xi \in \mathbb{C}$ and $\min\{|m'-\xi| : m' \in \mathbb{Z}\} \geq \delta > 0$. We therefore find (subject to δ and ν satisfying the hypotheses stated above (6.6.25)) that $|J^*_{\nu-p}(u)J^*_{\nu+p}(\overline{u})| \leq 4\delta^{-1}|\Gamma(\nu-p+1)\Gamma(\nu+p+1)|^{-1}\exp(|u|^2/2)$, and so (given the definition (1.9.5)) we obtain the result (6.6.25) ■

Lemma 6.6.8. *Let $\alpha \in (1/2,1)$, let $\varepsilon \in (0,1/2]$, and let $j \in \mathbb{N}$. Suppose moreover that $c \in \mathbb{C}^*$, and that $|c| \geq c_0 > 0$. Then, when $g \in G$ and $r = \rho(g)$, one has:*

$$\big(\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q}\kappa(\omega_1,\omega_2;c)\,\eta\big)(g) = \begin{cases} O_{\eta,\alpha,c_0,\omega_1,\omega_2,\epsilon}\left(|c|^{-2\alpha}r^{(1-\alpha)(1-\varepsilon)}\right) & \text{if } r \leq 1; \\ O_{\eta,\alpha,c_0,\omega_1,\omega_2,j}\left(|c|^{-2\alpha}r^{-j}\right) & \text{if } r \geq 1. \end{cases} \qquad (6.6.26)$$

Proof. Let $g \in G$, and let $r = \rho(g)$. If $r \leq 1$ then, by the result (6.6.23) of Lemma 6.6.6, the inequalities (6.6.24) and (6.6.25) of Lemma 6.6.7, the bound (6.5.15) of Lemma 6.5.3 (for $\omega' = \omega_2$, $\sigma_1 = 1$, $r_0 = |\omega_2|$ and

$d = 2\ell + 3$ (say), and with $(1-\alpha)\varepsilon$ substituted for ε) and the lower bound for $|\Gamma(\mu+1)|$ in (6.5.19), one finds (since $\epsilon(p) \leq 1$ for $p \in \mathbb{Z}$) that

$$\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\kappa(\omega_1,\omega_2;c)\,\eta\right)(g) \ll_{\ell,\omega_1,\omega_2,c_0,\alpha,\varepsilon} |u|^{2\alpha} r^{(1-\alpha)(1-\varepsilon)} \sum_{|p|\leq\ell} \int_{-\infty}^{\infty} |\eta(\alpha+it,p)| e^{(\pi/2)|t|}(1+|t|)^{\ell-\alpha-1/2}\mathrm{d}t \, +$$

$$+ \, |u|^2 r^{1-(1-\alpha)\varepsilon} \sum_{0<|p|\leq\ell} |\eta(0,p)| \; ,$$

where $u = 2\pi\sqrt{\omega_1\omega_2}/c$. By this, our hypotheses concerning α and c, and the conditions (T2) and (T3) stated below (6.4.3), the case $\rho(g) = r \leq 1$ of (6.6.26) follows. The other case of the result (6.6.26) may be proved similarly: for, when $\rho(g) = r \geq 1$, it follows by Lemma 6.6.6, Lemma 6.6.7 and Lemma 6.5.3 (with the result (6.5.15) of the latter being applied for $\omega' = \omega_2$, $\sigma_1 = (\alpha+1)/2$, $r_0 = |\omega_2|$ and $d = 2\ell+1+j$) that one has

$$\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\kappa(\omega_1,\omega_2;c)\,\eta\right)(g) \ll_{\ell,\omega_1,\omega_2,c_0,\alpha,j} |u|^{2\alpha} r^{-j-\alpha} \sum_{|p|\leq\ell} \int_{-\infty}^{\infty} |\eta(\alpha+it,p)| e^{(\pi/2)|t|}(1+|t|)^{3\ell+j-\alpha+1/2}\mathrm{d}t \, +$$

$$+ \, |u|^2 r^{-j} \sum_{0<|p|\leq\ell} |\eta(0,p)| \; ,$$

where, by hypothesis, $1/2 < \alpha < 1 < \sigma$ and $|u|^2 = |2\pi\sqrt{\omega_1\omega_2}/c|^2 \leq 4\pi^2|c_0|^{-2}|\omega_1\omega_2|$ ■

Lemma 6.6.9. *For each real $\beta > 3$, one has $L^{1+\beta}(N\backslash G) \ni |F_{\omega_2}^{\mathfrak{b}} P^{\mathfrak{a},*}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta|$.*

Proof. Put $\delta = [\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]^{-1}\delta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}$ (a complex constant), $f = \widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\eta$ and $F = F_{\omega_2}^{\mathfrak{b}} P^{\mathfrak{a},*}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta - \delta f$, so that by Lemma 6.6.4 one has

$$F(g) = \frac{\pi^2}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}\left(\omega_1,\omega_2;c\right)}{|c|^2} \left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\kappa\left(\omega_1,\omega_2;c\right)\eta\right)(g) \qquad (g \in G). \tag{6.6.27}$$

By Theorem 6.4.1 and Lemma 6.5.17, we have $\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\eta \in C^\infty(N\backslash G,\omega_2)$ and $P^{\mathfrak{a},*}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta \in C^\infty(\Gamma\backslash G)$. We consequently have $\{f, F+\delta f\} \subset C^\infty(N\backslash G,\omega_2)$, and so have also $|f|,|F|,|F+\delta f| \in C^0(N\backslash G,0)$ (the functions $|f|$, $|F|$ and $|F+\delta f|$ are, in particular, measurable). It therefore follows by the Hölder inequality $|F+\delta f|^{1+\beta} \leq (1+|\delta|^{1+1/\beta})^\beta(|F|^{1+\beta}+|f|^{1+\beta})$ and Lemma 6.6.5 that, for each $\beta \in (0,\infty)$ such that

$$\int_{N\backslash G} |F(g)|^{1+\beta}\mathrm{d}\dot{g} < \infty \; , \tag{6.6.28}$$

one has $\left|F_{\omega_2}^{\mathfrak{b}} P^{\mathfrak{a},*}\widetilde{\mathbf{L}}_{\ell,q}^{\omega_1}\eta\right| = |F+\delta f| = \left|F+\delta\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\eta\right| \in L^{1+\beta}(N\backslash G)$. Hence this proof will be complete once we have verified that the condition (6.6.28) is satisfied for all real $\beta > 3$.

Suppose now that β is a positive real number. Then, by (6.6.27), the bounds (6.6.26) of Lemma 6.6.8, the bound (6.5.59) of Lemma 6.5.9, the equation (6.1.25) and the result (6.1.26) of Lemma 6.1.5, it follows that when $g \in G$, $r = \rho(g)$ and $1/2 < \alpha < 1$ one has

$$F(g) = O_{\Gamma,|\omega_1|}\left(\zeta_{\mathbb{Q}(i)}^2\left(\alpha+\frac{1}{2}\right)\right) \cdot \begin{cases} O_{\eta,\alpha,\omega_1,\omega_2}\left(r^{(1-\alpha)(1-(\alpha-1/2))}\right) & \text{if } r \leq 1; \\ O_{\eta,\alpha,\omega_1,\omega_2}\left(r^{-1}\right) & \text{if } r \geq 1. \end{cases} \tag{6.6.29}$$

Given the definition of the measure $\mathrm{d}\dot{g}$ implicit in (6.2.9), it follows from (6.6.29) that when $\alpha \in (1/2,\infty)$ (so that $(1-\alpha)(1-(\alpha-1/2)) = (1/2)-(3/2)(\alpha-1/2)+(\alpha-1/2)^2 > (1/2)-3(\alpha-1/2) = 2-3\alpha$) one has:

$$\int_{N\backslash G} |F(g)|^{1+\beta}\mathrm{d}\dot{g} \ll_{\Gamma,\eta,\alpha,\beta,\omega_1,\omega_2} \int_0^1 \frac{r^{(2-3\alpha)(1+\beta)}\mathrm{d}r}{r^3} + \int_1^\infty \frac{r^{-(1+\beta)}\mathrm{d}r}{r^3} = \int_0^1 r^{(2-3\alpha)(1+\beta)-3}\mathrm{d}r + \frac{1}{\beta+3} \; . \tag{6.6.30}$$

Therefore the condition (6.6.28) is satisfied (for the given choice of $\beta \in (0,\infty)$) if there is some $\alpha > 1/2$ such that $(2-3\alpha)(1+\beta) > 2$. The latter is the case if and only if one has $(2-3\alpha)(1+\beta) > 2$ when $\alpha = 1/2$ ∎

Part I of the proof of Proposition 6.6.1: the geometric description of $\langle \phi_1, \phi_2 \rangle_{\Gamma\backslash G}$. By (6.6.1) and Lemma 6.6.2 (with $\mathfrak{b}$, θ and ω_2 substituted for $\mathfrak{a}$, η and ω, respectively), one obtains

$$\begin{aligned}[] [\Gamma_{\mathfrak{b}} : \Gamma'_{\mathfrak{b}}]\, \langle \phi_1, \phi_2 \rangle_{\Gamma\backslash G} &= \overline{[\Gamma_{\mathfrak{b}} : \Gamma'_{\mathfrak{b}}]\, \langle P^{\mathfrak{b},*} \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta, \phi_1 \rangle_{\Gamma\backslash G}} = \\ &= \overline{\langle \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta, F^{\mathfrak{b}}_{\omega_2} \phi_1 \rangle_{N\backslash G}} = \langle F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta \,, \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta \rangle_{N\backslash G} \ . \end{aligned} \tag{6.6.31}$$

This application of Lemma 6.6.2 is justified: for, given (6.6.1), it follows by Lemma 6.5.17 and (6.5.7) that one has $\phi_1 \in C^0(G) \cap L^2(\Gamma\backslash G)$, while by Lemma 6.6.9 one has (for example) $|F^{\mathfrak{b}}_{\omega_2} \phi_1| \in L^5(N\backslash G)$.

In preparation for an application of Lemma 6.6.4 we note that, by Lemma 6.4.2, the inner product $\langle \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \eta \,, \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta \rangle_{N\backslash G}$ exists (as an integral with respect to the measure $\mathrm{d}\dot{g}$ on $N\backslash G$) and one has, for any $\delta \in \mathbb{C}$,

$$\begin{aligned} &\langle F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta \,, \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta \rangle_{N\backslash G} - \frac{\delta}{\pi i} \sum_{p\in\mathbb{Z}} \int_{(0)} h_\ell(\nu,p) \left(p^2 - \nu^2\right) \mathrm{d}\nu = \\ &\qquad = \langle F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta \,, \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta \rangle_{N\backslash G} - \delta \, \langle \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \eta \,, \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta \rangle_{N\backslash G} = \langle F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta - \delta \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \eta \,, \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta \rangle_{N\backslash G} \ , \end{aligned}$$

where $h_\ell : \{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \le \sigma\} \times \mathbb{Z} \to \mathbb{C}$ is the function given by (6.6.3)-(6.6.4). In the above we may put $\delta = [\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]^{-1} \delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}$ (with $\delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2} \in \mathbb{C}$ as defined in (1.9.2)). Hence, given the result (6.6.15) of Lemma 6.6.4, we are able to deduce that

$$\begin{aligned} &\langle F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta \,, \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta \rangle_{N\backslash G} = \\ &\qquad = \frac{1}{\pi i \, [\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \, \delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2} \sum_{p\in\mathbb{Z}} \int_{(0)} h_\ell(\nu,p) \left(p^2 - \nu^2\right) \mathrm{d}\nu + \int_{N\backslash G} F(g) \, \overline{\left(\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta\right)(g)} \, \mathrm{d}\dot{g} \ , \end{aligned} \tag{6.6.32}$$

where $F = F^{\mathfrak{b}}_{\omega_2} P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q} \eta - [\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]^{-1} \delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2} \widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \eta$ is the very same function as occurs (within the proof of Lemma 6.6.9) on the left-hand side of the identity in (6.6.27).

In order to progress beyond (6.6.32) we must first show that the sum over $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ occurring on the right-hand side of the identity in (6.6.27) may be integrated over $N\backslash G$ term by term. We shall achieve this through an application of Lebesgue's theorem on 'dominated convergence', Theorem 1.34 of [40].

By the relation (6.1.26) of Lemma 6.1.5, the set ${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ is a countable subset of $\mathbb{C}^*$. Let c_* be any one-to-one function with domain $\mathbb{N}$ and range ${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$, and let $(F_M)_{M\in\mathbb{N}}$ be the sequence of functions on G given by:

$$F_M(g) = \frac{\pi^2}{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]} \sum_{m=1}^{M} \frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c_*(m))}{|c_*(m)|^2} \left(\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \kappa(\omega_1,\omega_2;c_*(m))\, \eta\right)(g) \qquad (M \in \mathbb{N},\ g \in G). \tag{6.6.33}$$

Given how (within the proof of Lemma 6.6.9) we obtained the estimate (6.6.29), it may be inferred that for all $g \in G$ the sum over $c \in {}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ in (6.6.27) is absolutely convergent, and that for any $M \in \mathbb{N}$ one may substitute $F_M(g)$ in place of $F(g)$ in (6.6.29). Therefore, and since (6.6.29) implies (6.6.30), we may deduce that for each $(\alpha,\beta) \in \mathbb{R}^2$ such that $1+\beta > 2/(2-3\alpha) > 4$ there exists some function $D_\alpha \in L^{1+\beta}(N\backslash G)$ satisfying the condition

$$D_\alpha(g) \ge |F_M(g)| \qquad \text{for all } M \in \mathbb{N},\ g \in G.$$

In particular (by the case $\alpha = 4/7$, $\beta = 7$ of the above), there must exist some function $D \in L^8(N\backslash G)$ such that $D : G \to [0,\infty)$ and

$$\left| F_M(g) \, \overline{\left(\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta\right)(g)} \right| \le \left| \left(\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta\right)(g) \right| D(g) \qquad \text{for all } M \in \mathbb{N},\ g \in G. \tag{6.6.34}$$

Since Lemma 6.6.5 implies that $|\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta| \in L^{8/7}(N\backslash G)$, it follows by the Hölder inequality of Section 12.42 of [43] that the above function D will, since it lies in $L^8(N\backslash G)$, be such that

$$|\widetilde{\mathbf{L}}^{\omega_2}_{\ell,q} \theta| \cdot D \in L^1(N\backslash G) \ . \tag{6.6.35}$$

By Lemma 6.4.3, and by Theorem 6.4.1 and Lemma 6.4.2 (applied with $\kappa(\omega_1,\omega_2;c)\eta$ substituted for η),

$$\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\kappa\left(\omega_1,\omega_2;c\right)\eta\right)\cdot\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)}\in L^1(N\backslash G)\qquad\text{for }\ c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}\subset\mathbb{C}^*. \tag{6.6.36}$$

Since the sum over $c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ in (6.6.27) is absolutely convergent for all $g\in G$, it follows by the definition (6.6.33) and our hypotheses concerning the function c_* that, for each $g\in G$,

$$F(g)\,\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)(g)}=\lim_{M\to\infty}F_M(g)\,\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)(g)}=\lim_{M\to\infty}\left(F_M\cdot\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)}\right)(g)\qquad\text{(say)},$$

and that the relations in (6.6.36) imply the relation $L^1(N\backslash G)\supseteq\{F_M\cdot\overline{(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta)}\,:M\in\mathbb{N}\}$. Therefore, given that we have also (6.6.34) and (6.6.35), it follows by Lebesgue's theorem on dominated convergence that

$$\lim_{M\to\infty}\int_{N\backslash G}F_M(g)\,\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)(g)}\,\mathrm{d}\dot{g}=\int_{N\backslash G}F(g)\,\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)(g)}\,\mathrm{d}\dot{g}\,. \tag{6.6.37}$$

Equation (6.6.37) enables us to justify term by term integration (over $N\backslash G$) of the sum over $c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$ seen in (6.6.27). Indeed, by the definitions (6.6.33) and (6.2.9) and the relations in (6.6.36), and by Lemma 6.4.3 and Lemma 6.4.2, we find that for $M\in\mathbb{N}$ one has

$$\begin{aligned}
&\frac{[\Gamma_{\mathfrak{a}}:\Gamma_{\mathfrak{a}}']}{4\pi^2}\int_{N\backslash G}F_M(g)\,\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)(g)}\,\mathrm{d}\dot{g}=\\
&\quad=\left\langle\frac{1}{4}\sum_{m=1}^{M}\frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c_*(m))}{|c_*(m)|^2}\,\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\kappa(\omega_1,\omega_2;c_*(m))\,\eta\,,\,\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right\rangle_{N\backslash G}=\\
&\quad=\frac{1}{4}\sum_{m=1}^{M}\frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c_*(m))}{|c_*(m)|^2}\left\langle\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\kappa(\omega_1,\omega_2;c_*(m))\,\eta\,,\,\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right\rangle_{N\backslash G}=\\
&\quad=\frac{1}{4\pi i}\sum_{m=1}^{M}\frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c_*(m))}{|c_*(m)|^2}\sum_{p=-\ell}^{\ell}\int_{(0)}\big(\kappa\big(\omega_1,\omega_2;c_*(m)\big)\eta\big)(\nu,p)\,\overline{\theta(\nu,p)}\,\lambda_\ell^*(\nu,p)\left(p^2-\nu^2\right)\mathrm{d}\nu=\\
&\quad=\sum_{m=1}^{M}\frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c_*(m))}{|c_*(m)|^2}\,(\mathbf{B}h_\ell)\!\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c_*(m)}\right),
\end{aligned}$$

where $h_\ell(\nu,p)$, $\lambda_\ell^*(\nu,p)$ and the $\mathbf{B}$-transform are defined as in (6.6.3), (6.6.4) and (1.9.3)-(1.9.6); it therefore follows by Equation (6.6.37) that

$$\sum_{m=1}^{\infty}\frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c_*(m))}{|c_*(m)|^2}\,(\mathbf{B}h_\ell)\!\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c_*(m)}\right)=\frac{[\Gamma_{\mathfrak{a}}:\Gamma_{\mathfrak{a}}']}{4\pi^2}\int_{N\backslash G}F(g)\,\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)(g)}\,\mathrm{d}\dot{g}\,. \tag{6.6.38}$$

The right-hand side of Equation (6.6.38) is independent of the choice of c_*. Moreover, given any permutation Λ on $\mathbb{N}$, we may substitute $c_*\circ\Lambda$ for c_* in the above: following any such substitution the function c_* will still be a one-to-one function with domain $\mathbb{N}$ and range ${}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}$, and so the result (6.6.38) will remain valid. Consequently, in light of Riemann's theorem (Theorem 8.33 of [1]) on conditionally convergent series, it may be deduced that the series on the left-hand side of Equation (6.6.38) is absolutely convergent. The equation (6.6.38) therefore has the equivalent formulation

$$\frac{[\Gamma_{\mathfrak{a}}:\Gamma_{\mathfrak{a}}']}{4\pi^2}\int_{N\backslash G}F(g)\,\overline{\left(\widetilde{\mathbf{L}}_{\ell,q}^{\omega_2}\theta\right)(g)}\,\mathrm{d}\dot{g}=\sum_{c\in{}^{\mathfrak{a}}\mathcal{C}^{\mathfrak{b}}}\frac{S_{\mathfrak{a},\mathfrak{b}}(\omega_1,\omega_2;c)}{|c|^2}\,(\mathbf{B}h_\ell)\!\left(\frac{2\pi\sqrt{\omega_1\omega_2}}{c}\right)\,. \tag{6.6.39}$$

By (6.6.31), (6.6.32) and (6.6.39) we obtain the final equality in (6.6.2), which is the desired geometric description of $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$ $\square$

Part II of the proof of Proposition 6.6.1: the spectral description of $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$. The first equality of (6.6.2) remains to be proved. In the above we obtained the geometric description of $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$ by applying Lemma 6.6.2 directly to $\langle\phi_2,\phi_1\rangle_{\Gamma\backslash G}=\overline{\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}}$ (see, in particular, (6.6.31)). In obtaining the spectral description of $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$ we shall apply Theorem A for $f_1=\phi_1$, $f_2=\phi_2$, and, rather than applying Lemma 6.6.2 directly to $\langle\phi_2,\phi_1\rangle_{\Gamma\backslash G}$ (or even to $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$), we shall instead apply it to each of the terms occurring, when $f_1=\phi_1$ and $f_2=\phi_2$, on the right-hand side of the Parseval identity (1.8.8). These applications of Lemma 6.6.2 are our first concern in what follows (the application of the Parseval identity is to be discussed later). Accordingly, we suppose now that

$$\phi\in\left\{1\,,\,T_V\varphi_{\ell,q}\left(\nu_V,p_V\right)\,,\,E^{\mathfrak{c}}_{\ell,q}\left(it_*,p_*\right)\right\},$$

where V is any one of the irreducible 'cuspidal' subspaces of $L^2(\Gamma\backslash G)$ occurring as a factor in the direct sum in (1.7.4) (ν_V,p_V being the associated spectral parameters), while $\mathfrak{c}$ is a cusp contained in some given set of representatives $\mathfrak{C}(\Gamma)$ of the Γ-equivalence classes of cusps, the Eisenstein series $E^{\mathfrak{c}}_{\ell,q}(\nu,p)$ is as defined in Subsection 1.8, and one has $t_*\in\mathbb{R}$ and $p_*\in[-\ell,\ell]\cap(1/2)[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]\mathbb{Z}$.

In applying Lemma 6.6.2 to $\langle\phi_1,\phi\rangle_{\Gamma\backslash G}=\langle P^{\mathfrak{a},*}\widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta,\phi\rangle_{\Gamma\backslash G}$ we are led to consider the term $F^{\mathfrak{a}}_{\omega_1}\phi$, from the Fourier expansion of ϕ at the cusp $\mathfrak{a}$. With regard to the case $\phi=1$, one finds by (1.4.2)-(1.4.3) that

$$F^{\mathfrak{a}}_{\omega_1}1=0\,. \tag{6.6.40}$$

Considering next the case $\phi=T_V\varphi_{\ell,q}(\nu_V,p_V)$ we observe that, in the discussion around (1.7.10)-(1.7.13), it was implicitly found that $F^{\mathfrak{a}}_{\omega_1}T_V\varphi_{\ell,q}(\nu_V,p_V)=c^{\mathfrak{a}}_V(\omega_1)\mathbf{J}_{\omega_1}\varphi_{\ell,q}(\nu_V,p_V)$; in terms of the modified Fourier coefficients defined in (1.7.15), this result becomes:

$$F^{\mathfrak{a}}_{\omega_1}T_V\varphi_{\ell,q}(\nu_V,p_V)=C^{\mathfrak{a}}_V\left(\omega_1;\nu_V,p_V\right)\left(\pi\left|\omega_1\right|\right)^{-\nu_V}\left(\omega_1/\left|\omega_1\right|\right)^{p_V}\mathbf{J}_{\omega_1}\varphi_{\ell,q}(\nu_V,p_V)\,. \tag{6.6.41}$$

Similarly, by what is noted in the discussion around (1.8.4)-(1.8.7) it may be inferred that, for all $(\nu,g)\in\mathbb{C}\times G$ such that $\mathrm{Re}(\nu)\geq 0$, one has $F^{\mathfrak{a}}_{\omega_1}E^{\mathfrak{c}}_{\ell,q}(\nu,p)=[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]^{-1}D^{\mathfrak{a}}_{\mathfrak{c}}(\omega_1;\nu,p)\mathbf{J}_{\omega_1}\varphi_{\ell,q}(\nu,p)$; in particular, one has

$$F^{\mathfrak{a}}_{\omega_1}E^{\mathfrak{c}}_{\ell,q}\left(it_*,p_*\right)=\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]^{-1}\left(\pi\left|\omega_1\right|\right)^{-it_*}\left(\omega_1/\left|\omega_1\right|\right)^{p_*}B^{\mathfrak{a}}_{\mathfrak{c}}\left(\omega_1;it_*,p_*\right)\mathbf{J}_{\omega_1}\varphi_{\ell,q}\left(it_*,p_*\right)\,, \tag{6.6.42}$$

where the 'modified Fourier coefficient' $B^{\mathfrak{a}}_{\mathfrak{c}}(\omega;\nu,p)$ is defined as Equation (1.8.9) would indicate.

By a calculation somewhat similar to that which yields the bound (6.6.30), it may be deduced from the estimates (6.5.15) of Lemma 6.5.3 (and from (1.5.16)) that, for $(\nu,p)\in\mathbb{C}\times\{-\ell,1-\ell,\ldots,\ell\}$ and $\beta\in(1,\infty)$, one has:

$$L^{1+\beta}(N\backslash G)\ni\left|\mathbf{J}_{\omega_1}\varphi_{\ell,q}(\nu,p)\right|\qquad\text{if }\;|\mathrm{Re}(\nu)|<1-2/(1+\beta). \tag{6.6.43}$$

By points noted in Subsection 1.7 (see in particular the paragraphs containing (1.7.5)-(1.7.8) and (1.7.14)), we have either $(\nu_V,p_V)\in(i\mathbb{R})\times\{-\ell,1-\ell,\ldots,\ell\}$, or else $p_V=0$ and $0\neq\nu_V\in(-1,1)$. In the former 'principal series' case, it follows by (6.6.41) and (6.6.43) that one has $L^{1+\beta}(N\backslash G)\ni|F^{\mathfrak{a}}_{\omega_1}T_V\varphi_{\ell,q}(\nu_V,p_V)|$ for all $\beta\in(1,\infty)$; in the latter 'complementary series' case it follows by the same results that, for all real $\beta>(1+|\nu_V|)/(1-|\nu_V|)$, one has $L^{1+\beta}(N\backslash G)\ni|F^{\mathfrak{a}}_{\omega_1}T_V\varphi_{\ell,q}(\nu_V,p_V)|$. By (6.6.42) and (6.6.43), one has also $L^{1+\beta}(N\backslash G)\ni|F^{\mathfrak{a}}_{\omega_1}E^{\mathfrak{c}}_{\ell,q}(it_*,p_*)|$ for all $\beta\in(1,\infty)$. Given the identity (6.6.40), and given the content of the last few observations (subsequent to (6.6.43)), it is certainly the case that

$$L^{1+\beta}(N\backslash G)\ni\left|F^{\mathfrak{a}}_{\omega_1}\phi\right|\qquad\text{for some }\;\beta=\beta(\phi)\in(0,\infty). \tag{6.6.44}$$

By (6.6.44), the function ϕ will satisfy all the relevant hypotheses of Lemma 6.6.2 if, for some $\alpha\in(0,\infty)$, it is contained in $C^0(G)\cap L^{1+\alpha}(\Gamma\backslash G)$. If $\phi=1$, then it is trivially the case that one has $\phi\in C^0(G)\cap L^\infty(\Gamma\backslash G)$. If $\phi=T_V\varphi_{\ell,q}(\nu_V,p_V)$ then by (1.7.3), (1.7.7), (1.7.8), (1.7.10) and the definitions (1.4.4)-(1.4.7) one has $\phi\in C^\infty(G)\cap{}^0L^2(\Gamma\backslash G)\subset C^0(G)\cap L^2(\Gamma\backslash G)$: in fact, by (1.7.10) and the bound (1.4.13) on the growth of any cusp form $f\in A^0_\Gamma(\Upsilon_{\nu,p};\ell,q)$, it follows that the $T_V\varphi_{\ell,q}(\nu_V,p_V)$ is a bounded function on G, so that one has also $\phi=T_V\varphi_{\ell,q}(\nu_V,p_V)\in L^\infty(\Gamma\backslash G)$. If $\phi=E^{\mathfrak{c}}_{\ell,q}(it_*,p_*)$ then, by (1.3.2) and the 'cusp-sector estimate' (6.2.20) of Lemma 6.2.8, it follows that there exists some $r_0=r_0(\Gamma,\ell,t_*)\in[1,\infty)$ such that, for all

$\mathfrak{d} \in \mathbb{Q}(i) \cup \{\infty\}$, and all $g \in G$ such that $\rho(g) \geq r_0$, one has $\phi(g_{\mathfrak{d}} g) = O_{\Gamma,\ell,t_*}(\rho(g))$ (it being assumed here that the scaling matrix $g_{\mathfrak{d}} \in G$ is such that (1.1.16) and (1.1.20)-(1.1.21) hold when $\mathfrak{d}$ is substituted for $\mathfrak{c}$), and so in this case one finds (by computations similar to those seen in the proof of Corollary 6.2.10) that the space $L^{1+\alpha}(\Gamma\backslash G)$ contains ϕ whenever α satisfies $0 < \alpha < 1$. Moreover (as is asserted in Subsection 1.8) one has $E^{\mathfrak{c}}_{\ell,q}(it_*, p_*) \in C^\infty(\Gamma\backslash G)$. This follows, by arguments similar to those employed in the proof of Lemma 6.5.15, from the analytic continuations of the summands occurring on the right-hand side of Equation (1.8.4): note, in particular, that a bound such as the estimate (11.49) of [32] enables one to establish that the Fourier expansion of $E^{\mathfrak{c}}_{\ell,q}(it_*, p_*)$ at the cusp ∞ is uniformly convergent on any compact subset of G. Given (6.5.7), and given what has so far been ascertained in the present paragraph, we may add to (6.6.44) the conclusion that

$$\phi \in C^0(G) \cap L^{3/2}(\Gamma\backslash G) . \tag{6.6.45}$$

By (6.6.44) and (6.6.45), Lemma 6.6.2 applies when (as we assume) $\phi \in \{1, T_V\varphi_{\ell,q}(\nu_V, p_V), E^{\mathfrak{c}}_{\ell,q}(it_*, p_*)\}$, and so, bearing in mind (6.6.40)-(6.6.42), it follows by the case $\omega = \omega_1$ of Equation (6.6.5) that

$$\begin{aligned}
&[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]\, \langle \phi_1, \phi\rangle_{\Gamma\backslash G} = \\
&\quad = [\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]\, \langle P^{\mathfrak{a},*} \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta, \phi\rangle_{\Gamma\backslash G} = \\
&\quad = \langle \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta, F^{\mathfrak{a}}_{\omega_1}\phi\rangle_{N\backslash G} = \\
&\quad = \begin{cases}
0 & \text{if } \phi = 1; \\
\overline{C^{\mathfrak{a}}_V(\omega_1; \nu_V, p_V)}\, \left\langle \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta\,,\, |\pi\omega_1|^{-\nu_V} (\omega_1/|\omega_1|)^{p_V} \mathbf{J}_{\omega_1}\varphi_{\ell,q}(\nu_V, p_V)\right\rangle_{N\backslash G} & \text{if } \phi = T_V\varphi_{\ell,q}(\nu_v, p_V); \\
\dfrac{1}{[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]}\, \overline{B^{\mathfrak{a}}_{\mathfrak{c}}(\omega_1; it_*, p_*)}\, \left\langle \widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta\,,\, |\pi\omega_1|^{-it_*} (\omega_1/|\omega_1|)^{p_*} \mathbf{J}_{\omega_1}\varphi_{\ell,q}(it_*, p_*)\right\rangle_{N\backslash G} & \text{if } \phi = E^{\mathfrak{c}}_{\ell,q}(it_*, p_*).
\end{cases}
\end{aligned}$$

This result, when expressed in terms of the Lebedev transform operator $\mathbf{L}^{\omega_1}_{\ell,q}$ (defined in (6.4.2)) becomes:

$$\begin{aligned}
&[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]\, \langle \phi_1, \phi\rangle_{\Gamma\backslash G} = \\
&\quad = \begin{cases}
0 & \text{if } \phi = 1; \\
\overline{C^{\mathfrak{a}}_V(\omega_1; \nu_V, p_V)}\, \dfrac{(-i)^{p_V}\pi^2 \left\|\Phi^{\ell}_{p_V,q}\right\|_K}{\Gamma(\ell+1+\overline{\nu_V})} \left(\mathbf{L}^{\omega_1}_{\ell,q}\widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta\right)(-\overline{\nu_V}, p_V) & \text{if } \phi = T_V\varphi_{\ell,q}(\nu_v, p_V); \\
\dfrac{1}{[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]}\, \overline{B^{\mathfrak{a}}_{\mathfrak{c}}(\omega_1; it_*, p_*)}\, \dfrac{(-i)^{p_*}\pi^2 \left\|\Phi^{\ell}_{p_*,q}\right\|_K}{\Gamma\left(\ell+1+\overline{(it_*)}\right)} \left(\mathbf{L}^{\omega_1}_{\ell,q}\widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta\right)\left(-\overline{(it_*)}, p_*\right) & \text{if } \phi = E^{\mathfrak{c}}_{\ell,q}(it_*, p_*).
\end{cases}
\end{aligned}$$

Moreover, since $\eta \in \mathcal{T}^{\ell}_{\sigma}$, and since $\sigma \in (1,2)$, it follows by the results (6.4.7) and (6.4.10) of Theorem 6.4.1, and (1.6.5) and (1.6.6) (for $\ell' = \ell$, $q' = q$), that for $(\nu, p) \in ((i\mathbb{R}) \times \{-\ell, 1-\ell, \ldots, \ell\}) \cup ((-1,1) \times \{0\})$ one has:

$$\begin{aligned}
&\frac{(-i)^p\pi^2 \left\|\Phi^{\ell}_{p,q}\right\|_K}{\Gamma(\ell+1+\overline{\nu})} \left(\mathbf{L}^{\omega_1}_{\ell,q}\widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\eta\right)(-\overline{\nu}, p) = \\
&\qquad = (-2\pi)(-i)^p\, \frac{\sin(\pi\nu)}{(\pi\nu)}\, \frac{\nu^{1+\epsilon(p)}}{(\nu^2 - p^2)}\, \eta(-\overline{\nu}, p) \;\times \\
&\qquad\qquad \times \begin{cases}
\left\|\varphi_{\ell,q}(\nu,p)\right\|_{\mathrm{ps}} \Gamma(\ell+1+\nu) & \text{if } (\nu,p) \in (i\mathbb{R}) \times \{-\ell, 1-\ell, \ldots, \ell\}; \\
\left\|\varphi_{\ell,q}(\nu,0)\right\|_{\mathrm{cs}} \sqrt{\Gamma(\ell+1+\nu)\Gamma(\ell+1-\nu)} & \text{if } (0,0) \neq (\nu,p) \in (-1,1) \times \{0\}.
\end{cases}
\end{aligned}$$

We therefore find that

$$\frac{[\Gamma_{\mathfrak{a}} : \Gamma'_{\mathfrak{a}}]}{(-2\pi)}\, \langle \phi_1, \phi\rangle_{\Gamma\backslash G} = 0 \qquad \text{if } \phi = 1, \tag{6.6.46}$$

whereas if $\phi = T_V\varphi_{\ell,q}(\nu_V, p_V)$ then (when the norm $\|T_V\varphi_{\ell,q}(\nu_V, p_V)\|_{\Gamma\backslash G}$ is defined as in (1.7.14)) one has

$$\begin{aligned}\frac{[\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]}{(-2\pi)}\langle\phi_1,\phi\rangle_{\Gamma\backslash G} &= (-i)^{p_V}\,\|T_V\varphi_{\ell,q}(\nu_V,p_V)\|_{\Gamma\backslash G}\,\overline{C_V^{\mathfrak{a}}(\omega_1;\nu_V,p_V)}\,\frac{\sin(\pi\nu_V)}{(\pi\nu_V)}\,\frac{\nu_V^{1+\epsilon(p_V)}}{(\nu_V^2-p_V^2)}\,\times\\ &\quad\times\eta\left(-\overline{\nu_V},p_V\right)\cdot\begin{cases}\Gamma(\ell+1+\nu_V) & \text{if } \nu_V^2\le 0,\\ \sqrt{\Gamma(\ell+1+\nu_V)\,\Gamma(\ell+1-\nu_V)} & \text{if } 1>\nu_V^2>0=p_V,\end{cases}\end{aligned} \tag{6.6.47}$$

and if it is instead the case that $\phi = E^{\mathfrak{c}}_{\ell,q}(it_*, p_*)$ then

$$\begin{aligned}&\frac{[\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]}{(-2\pi)}\langle\phi_1,\phi\rangle_{\Gamma\backslash G} =\\ &\quad= \frac{(-i)^{p_*}\,\|\varphi_{\ell,q}(it_*,p_*)\|_{\mathrm{ps}}}{[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]}\,\overline{B_{\mathfrak{c}}^{\mathfrak{a}}(\omega_1;it_*,p_*)}\,\frac{\sin(\pi it_*)}{(\pi it_*)}\,\frac{(it_*)^{1+\epsilon(p_*)}}{\left((it_*)^2-p_*^2\right)}\,\eta\big(-\overline{(it_*)},p_*\big)\Gamma(\ell+1+it_*)\,.\end{aligned} \tag{6.6.48}$$

By the symmetry apparent in our hypothesis (6.6.1), formulae corresponding to the above may be obtained for $(-2\pi)^{-1}[\Gamma_{\mathfrak{b}}:\Gamma'_{\mathfrak{b}}]\langle\phi_2,\phi\rangle_{\Gamma\backslash G} = (-2\pi)^{-1}[\Gamma_{\mathfrak{b}}:\Gamma'_{\mathfrak{b}}]\langle P^{\mathfrak{b},*}\widetilde{\mathbf{L}}^{\omega_1}_{\ell,q}\theta,\phi\rangle_{\Gamma\backslash G}$. Consequently, if we put

$$F_{\phi_1,\phi_2}(\phi) = \frac{[\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]\,[\Gamma_{\mathfrak{b}}:\Gamma'_{\mathfrak{b}}]}{4\pi^2}\langle\phi_1,\phi\rangle_{\Gamma\backslash G}\,\langle\phi,\phi_2\rangle_{\Gamma\backslash G} = \overline{\left(\frac{[\Gamma_{\mathfrak{b}}:\Gamma'_{\mathfrak{b}}]}{(-2\pi)}\langle\phi_2,\phi\rangle_{\Gamma\backslash G}\right)}\,\frac{[\Gamma_{\mathfrak{a}}:\Gamma'_{\mathfrak{a}}]}{(-2\pi)}\langle\phi_1,\phi\rangle_{\Gamma\backslash G}\ , \tag{6.6.49}$$

and take $h_\ell : \{\nu\in\mathbb{C} : |\mathrm{Re}(\nu)|\le\sigma\}\times\mathbb{Z}\to\mathbb{C}$ to be given by (6.6.3) and (6.6.4), then, by (6.6.46), (6.6.47) and (6.6.48), and the corresponding formulae for $(-2\pi)^{-1}[\Gamma_{\mathfrak{b}}:\Gamma'_{\mathfrak{b}}]\langle\phi_2,\phi\rangle_{\Gamma\backslash G}$, it follows that

$$F_{\phi_1,\phi_2}(\phi) = \begin{cases} 0 & \text{if } \phi = 1;\\ \|T_V\varphi_{\ell,q}(\nu_V,p_V)\|^2_{\Gamma\backslash G}\,\overline{C_V^{\mathfrak{a}}(\omega_1;\nu_V,p_V)}\,C_V^{\mathfrak{b}}(\omega_2;\nu_V,p_V)\,h_\ell(\nu_V,p_V) & \text{if } \phi = T_V\varphi_{\ell,q}(\nu_V,p_V);\\ \dfrac{\|\varphi_{\ell,q}(it_*,p_*)\|^2_{\mathrm{ps}}}{[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]^2}\,\overline{B_{\mathfrak{c}}^{\mathfrak{a}}(\omega_1;it_*,p_*)}\,B_{\mathfrak{c}}^{\mathfrak{b}}(\omega_2;it_*,p_*)\,h_\ell(it_*,p_*) & \text{if } \phi = E^{\mathfrak{c}}_{\ell,q}(it_*,p_*).\end{cases} \tag{6.6.50}$$

In arriving at the results stated in (6.6.50) we have used both the fact that for $\mathrm{Re}(\nu) > -1$ one has $\Gamma(\ell+1+\overline{\nu}) = \overline{\Gamma(\ell+1+\nu)}$, and the fact that the function $\alpha : \mathbb{C}\times\mathbb{Z}\to\mathbb{C}$ given by

$$\alpha(\nu,p) = \frac{\sin(\pi\nu)}{(\pi\nu)}\,\frac{\nu^{1+\epsilon(p)}}{(\nu^2-p^2)} = \frac{(-1)^p}{\Gamma(1+\nu+|p|)\,\Gamma(1-\nu+|p|)}\prod_{1\le m<|p|}\left(\nu^2-m^2\right) \qquad (\nu\in\mathbb{C},\ p\in\mathbb{Z})$$

satisfies both $\alpha(\overline{\nu},p) = \overline{\alpha(\nu,p)}$ and $\alpha(-\nu,p) = \alpha(\nu,p)$, for all $(\nu,p)\in\mathbb{C}\times\mathbb{Z}$. We have also made use of both the observation that if $(\nu,p)\in\mathbb{C}\times\mathbb{Z}$ and $\nu^2\le 0$ then $\overline{\nu} = -\nu$, and the (complementary) observation that if it is instead the case that $p=0$ and $1>\nu^2>0$ then $-\overline{\nu} = -\nu$, $p=-p$, $\sqrt{\Gamma(\ell+1+\nu)\,\Gamma(\ell+1-\nu)}\in\mathbb{R}$ and $\eta(-\nu,-p) = \eta(\nu,p)$ (the last equality following by virtue of the condition (T1) stated below (6.4.3), and our hypothesis that $\eta\in\mathcal{T}_\sigma^\ell$).

Observe now that, by (6.6.1), Lemma 6.5.17 and (6.5.7), the hypotheses of Theorem A are satisfied when one has (there) $f_1=\phi_1$ and $f_2=\phi_2$. It is therefore implied by Theorem A that, for $f_1=\phi_1$ and $f_2=\phi_2$, the 'Parseval identity' stated in (1.8.8) is valid. By the case $f_1=\phi_1$, $f_2=\phi_2$ of (1.8.8), combined with (6.6.49)-(6.6.50), we obtain the first inequality in (6.6.2), which is the desired spectral description of $\langle\phi_1,\phi_2\rangle_{\Gamma\backslash G}$. The sums and integrals appearing on the left-hand side of the first equality in (6.6.2) are simply an alternative formulation of the sums and integrals which (in the case $f_1=\phi_1$, $f_2=\phi_2$) appear on the

right-hand side of Equation (1.8.8); it is therefore a corollary of Theorem A that these sums and integrals are absolutely convergent. This completes our proof of the preliminary sum formula, Proposition 6.6.1 ■

§6.7 Completing the proof of the spectral summation formula.

In this final subsection of our Appendix we show that the preliminary sum formula (Proposition 6.6.1) implies the more general result asserted in Theorem B. Our proof of this is closely modelled on the 'Extension Method' employed by Lokvenec-Guleska in [32], Subsection 11.2 and Subsection 11.3.

It is to be assumed henceforth that ω_1, ω_2, $\mathfrak{a}$, $\mathfrak{b}$, $g_{\mathfrak{a}}$ and $g_{\mathfrak{b}}$ are given, with $0 \neq \omega_1, \omega_2 \in \mathfrak{O}$, and with $\mathfrak{a}, \mathfrak{b} \in \mathbb{Q}(i) \cup \{\infty\}$ and $g_{\mathfrak{b}}, g_{\mathfrak{b}} \in G$ such that (1.1.16) and (1.1.20)-(1.1.21) hold for $\mathfrak{c} \in \{\mathfrak{a}, \mathfrak{b}\}$. We begin our implementation of Lokvenec-Guleska's 'Extension Method' by defining some of the relevant terminology.

When $\sigma \in (0, \infty)$ and $\varrho, \vartheta \in \mathbb{R}$, let $\mathcal{H}_0^\sigma(\varrho, \vartheta)$ be the space of functions $h : \{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{Z} \to \mathbb{C}$ satisfying, for the given choice of σ, ϱ and ϑ, the conditions (i)-(iii) of Theorem B.

For $\sigma \in (1/2, 1)$, $\varrho \in (2, \infty)$ and $\vartheta \in (3, \infty)$, we define $\mathcal{H}_\star^\sigma(\varrho, \vartheta)$ to be equal to the space $\mathcal{H}_0^\sigma(\varrho, \vartheta)$; for $\sigma \in (1, 2)$, $\varrho \in (2, \infty)$ and $\vartheta \in (3, \infty)$, we define $\mathcal{H}_\star^\sigma(\varrho, \vartheta)$ to be the subspace of $\mathcal{H}_0^\sigma(\varrho, \vartheta)$ containing just those members h of $\mathcal{H}_0^\sigma(\varrho, \vartheta)$ that (in addition to satisfying the conditions (i)-(iii) of Theorem B) are such that, for each $p \in \mathbb{Z}$, the function $\nu \mapsto h(\nu, p)$ has a zero of order at least 2 at the point $\nu = 1$.

For $\sigma \in (1/2, 1) \cup (1, 2)$, $\varrho, \vartheta \in (3, \infty)$, $h \in \mathcal{H}_\star^\sigma(\varrho, \vartheta)$, $\omega, \omega' \in \{\omega_1, \omega_2\}$ and $\mathfrak{d}, \mathfrak{d}' \in \{\mathfrak{a}, \mathfrak{b}\}$, we put

$$\chi_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h) = \frac{1}{4\pi^3 i} \, \delta_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'} \sum_{p \in \mathbb{Z}} \int_{(0)} h(\nu, p) \left(p^2 - \nu^2\right) \mathrm{d}\nu \, , \tag{6.7.1}$$

$$X_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h) = \sum_{c \in {}^{\mathfrak{d}}\mathcal{C}^{\mathfrak{d}'}} \frac{S_{\mathfrak{d},\mathfrak{d}'}(\omega, \omega'; c)}{|c|^2} \, (\mathbf{B}h)\!\left(\frac{2\pi\sqrt{\omega\omega'}}{c}\right) \tag{6.7.2}$$

(with $\mathbf{B}h : \mathbb{C}^* \to \mathbb{C}$ as defined in (1.9.3)-(1.9.6)) and, subject to the absolute convergence of the relevant sums and integrals,

$$\begin{aligned} Y_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h) = &\sum_V \overline{C_V^{\mathfrak{d}}(\omega; \nu_V, p_V)} \, C_V^{\mathfrak{d}'}(\omega'; \nu_V, p_V) \, h(\nu_V, p_V) \, + \\ &+ \sum_{\mathfrak{c} \in \mathfrak{C}(\Gamma)} \frac{1}{4\pi i \, [\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]} \sum_{p \in \frac{1}{2}[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]\mathbb{Z}} \int_{(0)} \overline{B_{\mathfrak{c}}^{\mathfrak{d}}(\omega; \nu, p)} \, B_{\mathfrak{c}}^{\mathfrak{d}'}(\omega'; \nu, p) \, h(\nu, p) \, \mathrm{d}\nu \, . \end{aligned} \tag{6.7.3}$$

By the phrase 'the sum formula for $Y_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h)$ is valid', we shall mean that all sums and integrals on the right-hand side of Equation (6.7.3) are absolutely convergent, and that one has

$$Y_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h) = \chi_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h) + X_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h) \, . \tag{6.7.4}$$

Remark 6.7.1. Let σ, ϱ, ϑ and h satisfy the hypotheses assumed in defining $\chi_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h)$, $X_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h)$ and $Y_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h)$. Then $\varrho > 3$, $\vartheta > 3$ and h, σ, ϱ and ϑ are, in particular, such that the conditions (ii) and (iii) of Theorem B are satisfied; therefore the integrals and sum on the right-hand side of Equation (6.7.1) are absolutely convergent. It moreover follows by [32], Lemma 11.1.1 and Lemma 11.1.2, that the transform $\mathbf{B}h : \mathbb{C}^* \to \mathbb{C}$ is a well-defined function (by virtue of the integrals and sum on the right-hand side of Equation (1.9.3) being absolutely convergent for all $u \in \mathbb{C}^*$), and that this transform $\mathbf{B}h$ satisfies

$$\sup\left\{ |u|^{-2\min\{\sigma,1\}} |(\mathbf{B}h)(u)| \, : \, u \in \mathbb{C} \text{ and } 0 < |u| \leq r_1 \right\} < \infty \qquad \text{for some } r_1 \in (0, \infty). \tag{6.7.5}$$

Given our assumption of the hypothesis that $\sigma \in (1/2, 1) \cup (1, 2)$, it follows by (6.7.5), Lemma 6.5.9 and the result (6.1.26) of Lemma 6.1.5 that the sum over $c \in {}^{\mathfrak{d}}\mathcal{C}^{\mathfrak{d}'}$ occurring in (6.7.2) is absolutely convergent. Note too that by Remark 1.9.2 it follows that when $h \in \mathcal{H}_\star^\sigma(\varrho, \vartheta)$ and $\sigma > 1/2$ it is then the case that all the summands of the first sum on the right-hand side of Equation (6.7.3) are defined. The absolute convergence

of all the sums and integrals occurring on the right-hand side of Equation (6.7.3) will be established later on, in the course of our proof of Theorem B; in the meantime it cannot be taken for granted.

We divide our proof of Theorem B into two principal stages. In the first of these stages we deduce from Proposition 6.6.1 (the Preliminary Sum Formula) the following result.

Proposition 6.7.2 (weak sum formula). *Let $\sigma \in (1,2)$, let $\varrho, \vartheta \in (3,\infty)$, and let $f \in \mathcal{H}_\star^\sigma(\varrho,\vartheta)$. Then the sum formula for $Y_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}(f)$ is valid.*

Our proof of Proposition 6.7.2 is modelled on the initial steps in Lokvenec-Guleska's 'Extension Method' of [32], Subsection 11.2 and Subsection 11.3; we approach it via nine lemmas, two of which (Lemma 6.7.9 and Lemma 6.7.11) are used again in the final stage of our proof of Theorem B. We omit the proofs of Lemma 6.7.7 and Lemma 6.7.8, which differ from the proofs of Lemma 6.7.4 and Lemma 6.7.5 (respectively) only in that they involve the application of Lebesgue's 'dominated convergence' theorem (Theorem 1.34 of [40]), whereas it is Lebesgue's 'monotone convergence' theorem that is applied in the proofs of Lemma 6.7.4 and Lemma 6.7.5.

Lemma 6.7.3. *Let $\sigma \in (0,1) \cup (1,2)$, $\varrho \in (2,\infty)$ and $\vartheta \in (3,\infty)$, let $f \in \mathcal{H}_\star^\sigma(\varrho,\vartheta)$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{H}_\star^\sigma(\varrho,\vartheta)$ converging pointwise to f and satisfying*

$$\sup\left\{(1+|\mathrm{Im}(\nu)|)^\varrho\,(1+|p|)^\vartheta\,|f_n(\nu,p)| \,:\, n\in\mathbb{N},\ p\in\mathbb{Z},\ \nu\in\mathbb{C}\ \text{and}\ \mathrm{Re}(\nu)=\sigma\right\} < \infty\ . \qquad (6.7.6)$$

Then the integrals and sum defining the transform $\mathbf{B}f$ are absolutely convergent: each of the transforms in the sequence $\mathbf{B}f, \mathbf{B}f_1, \mathbf{B}f_2, \mathbf{B}f_3 \ldots$ is a complex valued function with domain $\mathbb{C}^*$. One has, moreover,

$$\lim_{n\to\infty}\left(\sup\left\{|u|^{-2\min\{\sigma,1\}}\,|(\mathbf{B}f_n - \mathbf{B}f)(u)| \,:\, u\in\mathbb{C}\ \text{and}\ 0<|u|\le r\right\}\right) = 0 \qquad \textit{for each } r\in(0,\infty). \quad (6.7.7)$$

Proof. This lemma is a minor refinement on Lemma 11.1.3 of [32]. Observe firstly that, since f and each function in the series $(f_n)_{n\in\mathbb{N}}$ lies in the space $\mathcal{H}_\star^\sigma(\varrho,\vartheta)$, and since we have (6.7.6), there therefore exists some $C_0 \in [0,\infty)$ such that

$$|(f_n - f)(\nu,p)| \le \frac{C_0}{(1+|\mathrm{Im}(\nu)|)^\varrho\,(1+|p|)^\vartheta} \qquad \text{for all } (\nu,p,n)\in\mathbb{C}\times\mathbb{Z}\times\mathbb{N} \text{ such that } |\mathrm{Re}(\nu)|=\sigma \quad (6.7.8)$$

(i.e. this follows since our hypotheses imply that the condition (iii) of Theorem B is satisfied when $h=f$, and that the condition (i) of Theorem B is satisfied when $h\in\{f\}\cup\{f_n : n\in\mathbb{N}\}$). Our hypotheses imply, moreover, that the conditions (ii) and (iii) of Theorem B are satisfied when $h\in\{f\}\cup\{f_n : n\in\mathbb{N}\}$ and σ, ϱ and ϑ are as given (with, in particular, $\varrho>2>0$, so that one has $(f_n-f)(\nu,p)\to 0$ as $|\mathrm{Im}(\nu)|\to\infty$ with n and p fixed and ν constrained to satisfy $|\mathrm{Re}(\nu)|\le\sigma$). Therefore, by an application of the maximum principle for analytic functions, one may deduce from (6.7.8) that, for all $n\in\mathbb{N}$ and all $p\in\mathbb{Z}$, one has

$$\sup\left\{\,|(f_n-f)(\nu,p)| \,:\, \nu\in\mathbb{C}\ \text{and}\ |\mathrm{Re}(\nu)|\le\sigma\right\} = \sup\left\{\,|(f_n-f)(\nu,p)| \,:\, \nu\in\mathbb{C}\ \text{and}\ |\mathrm{Re}(\nu)|=\sigma\right\} \le$$
$$\le C_0\,(1+|p|)^{-\vartheta}\ .$$

Hence, in addition to (6.7.8), one has $|(f_n-f)(0,p)|\le C_0(1+|p|)^{-\vartheta}$ for all $n\in\mathbb{N}$ and all $p\in\mathbb{Z}$. Therefore our hypothesis (6.7.6), despite being weaker than the corresponding hypothesis in Lemma 11.1.3 of [32], does in fact imply all of the assumptions that are relied upon in the (sketchy, but valid) proof supplied, in [32], for that lemma, and so (by the steps indicated on Page 100 of [32]) the result (6.7.7) follows. The results stated between (6.7.6) and (6.7.7) follow immediately from Lemma 11.1.1 of [32] ■

Lemma 6.7.4. *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of mappings from $\mathbb{N}$ into $\mathbb{R}\cup\{\infty\}$. Suppose moreover that*

$$0\le F_1(m)\le F_2(m)\le\ldots \qquad \textit{for each } m\in\mathbb{N}. \qquad (6.7.9)$$

Then, for some $\lambda \in \mathbb{R} \cup \{\infty\}$, one has

$$\sum_{m=1}^{\infty} \Bigl(\lim_{n\to\infty} F_n(m) \Bigr) = \lambda = \lim_{n\to\infty} \Biggl(\sum_{m=1}^{\infty} F_n(m) \Biggr) . \tag{6.7.10}$$

Proof. Let $\mathbf{P}(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$. Define the function $\mu_{\mathbb{Z}} : \mathbf{P}(\mathbb{N}) \to [0,\infty]$ by setting $\mu_{\mathbb{Z}}(A)$ equal to the cardinality of A whenever $\mathbb{N} \supseteq A$. Then $\mu_{\mathbb{Z}}$ is a positive measure on $\mathbb{Z}$, and for $n \in \mathbb{N}$ one has $\sum_{m=1}^{\infty} F_n(m) = \int_{\mathbb{Z}} F_n \mathrm{d}\mu_{\mathbb{Z}}$ (the expression on the right-hand side of this equation denoting the Lebesgue integral of F_n over $\mathbb{Z}$, with respect to the measure $\mu_{\mathbb{Z}}$). The result (6.7.10) is therefore equivalent to a special case of Lebesgue's 'monotone convergence' theorem, Theorem 1.26 of [40] ■

Lemma 6.7.5. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of mappings from $\mathbb{R} \times \mathbb{Z}$ into $\mathbb{R} \cup \{\infty\}$. Suppose moreover that*

$$0 \le \Phi_1(t,p) \le \Phi_2(t,p) \le \ldots \qquad \text{for each } (t,p) \in \mathbb{R} \times \mathbb{Z}, \tag{6.7.11}$$

and that, for all $n \in \mathbb{N}$ and all $p \in \mathbb{Z}$, the mapping $t \mapsto \Phi_n(t,p)$ is a function on $\mathbb{R}$ that is measurable with respect to the Lebesgue measure on $\mathbb{R}$. Then, for some $\lambda \in \mathbb{R} \cup \{\infty\}$, one has

$$\sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \Bigl(\lim_{n\to\infty} \Phi_n(t,p) \Bigr) \mathrm{d}t = \lambda = \lim_{n\to\infty} \Biggl(\sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_n(t,p) \, \mathrm{d}t \Biggr) . \tag{6.7.12}$$

Proof. For $n \in \mathbb{N}$, one has $\sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_n(t,p) \, \mathrm{d}t = \int_{\mathbb{R}\times\mathbb{Z}} \Phi_n \mathrm{d}\mu_{\mathbb{R}\times\mathbb{Z}}$, where $\mu_{\mathbb{R}\times\mathbb{Z}}$ is a positive measure on $\mathbb{R} \times \mathbb{Z}$ (defined so that, for all $p \in \mathbb{Z}$, all $M \in [0,\infty]$ and all $A \subseteq \mathbb{R}$ such that A has Lebesgue measure M, one has $\mu_{\mathbb{R}\times\mathbb{Z}}(A \times \{p\}) = M$). Therefore, like the lemma which preceded it, this lemma is equivalent to a special case of Lebesgue's 'monotone convergence' theorem ■

Lemma 6.7.6. *Let $\sigma \in (1/2,1) \cup (1,2)$, $\varrho, \vartheta \in (3,\infty)$, $\mathfrak{d} \in \{\mathfrak{a},\mathfrak{b}\}$, $\omega \in \{\omega_1,\omega_2\}$ and $f \in \mathcal{H}_{\star}^{\sigma}(\varrho,\vartheta)$. Suppose moreover that*

$$E = \bigl((i\mathbb{R}) \times \mathbb{Z} \bigr) \cup \bigl([-2/9, 2/9] \times \{0\} \bigr) , \tag{6.7.13}$$

and that $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{H}_{\star}^{\sigma}(\varrho,\vartheta)$ which satisfies the condition (6.7.6) of Lemma 6.7.3 and the following conditions:

(i) for all $n \in \mathbb{N}$, the sum formula for $Y_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f_n)$ is valid;

(ii) for all $(\nu,p) \in E$ one has $0 \le f_1(\nu,p) \le f_2(\nu,p) \le \ldots$;

(iii) for all $(\nu,p) \in \mathbb{C} \times \mathbb{Z}$ such that $|\mathrm{Re}(\nu)| \le \sigma$, one has $\lim_{n\to\infty} f_n(\nu,p) = f(\nu,p)$.

Then the sum formula for $Y_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f)$ is valid, and one has

$$\chi_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f) = \lim_{n\to\infty} \chi_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f_n) , \tag{6.7.14}$$

$$X_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f) = \lim_{n\to\infty} X_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f_n) , \tag{6.7.15}$$

$$Y_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f) = \lim_{n\to\infty} Y_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f_n) . \tag{6.7.16}$$

Proof. Given the definition of $\chi_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(h)$ in (6.7.1), and given (1.9.2) (where the notation '$\delta_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}$' is defined), it follows by (1.1.20) and Lemma 4.2 that, for $h \in \mathcal{H}_{\star}^{\sigma}(\varrho,\vartheta)$ we have

$$\chi_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(h) = \frac{1}{2\pi^3} \sum_{p\in\mathbb{Z}} \int_{-\infty}^{\infty} h(it,p) \left(t^2+p^2\right) \mathrm{d}t \qquad \text{and} \qquad \chi_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(h) \in \mathbb{C} \tag{6.7.17}$$

(the integrals and sum on the right-hand side of this equation being absolutely convergent, as discussed in our Remark 6.7.1). In particular, the hypotheses of the lemma suffice to ensure that (6.7.17) holds for all

$h \in \{f_n : n \in \mathbb{N}\} \cup \{f\}$. Therefore, since the form $t^2 + p^2$ is positive definite, and since (by hypothesis) the conditions (ii) and (iii) of the lemma are satisfied, it follows that (6.7.11) and all the other hypotheses of Lemma 6.7.5 are satisfied if $(\Phi_n)_{n\in\mathbb{N}} = (\Phi_n(t,p))_{n\in\mathbb{N}} = ((t^2+p^2)f_n(it,p))_{n\in\mathbb{N}}$. Hence the result (6.7.12) of Lemma 6.7.5 implies the equality in (6.7.14).

Given the definition of $X^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$ in (6.7.2) and what is observed in our Remark 6.7.1, and given our hypotheses concerning f and the sequence (f_n), it follows that

$$X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(h) \in \mathbb{C} \qquad \text{for all } \ h \in \{f_n : n \in \mathbb{N}\} \cup \{f\}, \tag{6.7.18}$$

and that, by virtue of the result (6.1.26) of Lemma 6.1.5, one has, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left|X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) - X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n)\right| \leq \\
&\qquad \leq \sum_{c \in {}^{\mathfrak{d}}\mathcal{C}^{\mathfrak{d}}} \frac{|S_{\mathfrak{d},\mathfrak{d}}(\omega,\omega;c)|}{|c|^2} \left|(\mathbf{B}f - \mathbf{B}f_n)\left(\frac{2\pi\omega}{c}\right)\right| \leq \\
&\qquad \leq |2\pi\omega|^{2\min\{1,\sigma\}} \left(\sup\left\{\frac{|(\mathbf{B}f - \mathbf{B}f_n)(u)|}{|u|^{2\min\{\sigma,1\}}} \,:\, u \in \mathbb{C} \text{ and } 0 < |u| \leq \frac{2\pi|\omega|}{|m_{\mathfrak{d}}|}\right\}\right) \sum_{c \in {}^{\mathfrak{d}}\mathcal{C}^{\mathfrak{d}}} \frac{|S_{\mathfrak{d},\mathfrak{d}}(\omega,\omega;c)|}{|c|^{2\min\{1+\sigma,2\}}} \,.
\end{aligned}$$

Therefore, since $\sigma > 1/2$, and since the sequence $(f_n)_{n\in\mathbb{N}}$ satisfies the relevant hypotheses, it follows by Lemma 6.5.9 and Lemma 6.7.3 that the equality in (6.7.15) must hold.

Now that (6.7.14) and (6.7.15) have been proved (and given that it therefore follows by (6.7.17) and (6.7.18) that the relevant sequences, $(\chi^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n))_{n\in\mathbb{N}}$ and $(X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n))_{n\in\mathbb{N}}$, are convergent sequences of complex numbers), it will suffice for the completion of the proof of the lemma that we show that the equality in (6.7.16) holds: for, when that is achieved, it may then be inferred from (6.7.14)-(6.7.16) that, since the condition (i) of the lemma is satisfied, one has

$$\chi^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) + X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) - Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) = \lim_{n\to\infty}\left(\chi^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n) + X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n) - Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n)\right) = \lim_{n\to\infty} 0 = 0\,, \tag{6.7.19}$$

so that the sum formula for $Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f)$ is valid (no proof of (6.7.16) being complete without it having been shown that all the sums and integrals occurring on the right-hand side of Equation (6.7.3) are absolutely convergent when $h = f$, $\mathfrak{d}' = \mathfrak{d}$ and $\omega' = \omega$).

Given that the condition (i) of the lemma is satisfied, it follows by (6.7.14), (6.7.15), (6.7.18) and the case $h = f$ of (6.7.17) that we have now

$$\mathbb{C} \ni \chi^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) + X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) = \lim_{n\to\infty}\left(\chi^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n) + X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n)\right) = \lim_{n\to\infty} Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n)\,. \tag{6.7.20}$$

Moreover, by the definition (6.7.3) of $Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$, we have also

$$\begin{aligned}
&Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(h) = \\
&\quad = \sum_V \left|C^{\mathfrak{d}}_V(\omega;\nu_V,p_V)\right|^2 h(\nu_V,p_V) + \sum_{\mathfrak{c}\in\mathfrak{C}(\Gamma)} \frac{1}{4\pi\,[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]} \sum_{p\in\frac{1}{2}[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]\mathbb{Z}} \int_{-\infty}^{\infty} \left|B^{\mathfrak{d}}_{\mathfrak{c}}(\omega;it,p)\right|^2 h(it,p)\,\mathrm{d}t\,,
\end{aligned} \tag{6.7.21}$$

for all $h \in \mathcal{H}^\sigma_\star(\varrho,\vartheta)$ such that the sums and integrals on the right-hand side of this equation are absolutely convergent. Since the conditions (i) and (ii) of the lemma are satisfied, and since each factor $|C^{\mathfrak{d}}_V(\omega;\nu_V,p_V)|$ and $\left|B^{\mathfrak{d}}_{\mathfrak{c}}(\omega;it,p)\right|$ occurring on the right-hand side of Equation (6.7.21) is real and non-negative, it follows that (6.7.21) holds for all $h \in \{f_n : n \in \mathbb{N}\}$, and that, given (6.7.13) and what is noted in Remark 1.9.2, one has $0 \leq Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_1) \leq Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_2) \leq \ldots$. Hence, and by (6.7.20), the complex number $\chi^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) + X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f)$ must be real and non-negative (certainly one can give a more direct proof that it is real), and one must have

$$\infty > \chi^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) + X^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f) \geq Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n) \geq 0 \qquad \text{for all } \ n \in \mathbb{N}. \tag{6.7.22}$$

In the first sum on the right-hand side of Equation (6.7.21) one sums over only countably many cuspidal irreducible subspaces V (i.e. just those occurring in the decomposition (1.7.4) of the space ${}^0L^2(\Gamma\backslash G)$). Let

the relevant subspaces V be arranged in a sequence, $V(1), V(2), \ldots$ (say). Then, since (6.7.21) holds for all $h \in \{f_n : n \in \mathbb{N}\}$, we have

$$Y_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f_n) = \sum_{m=1}^{\infty} F_n(m) + \sum_{\mathfrak{c}\in\mathfrak{C}(\Gamma)} \frac{1}{4\pi\,[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]} \sum_{P\in\mathbb{Z}} \int_{-\infty}^{\infty} \Phi_{\mathfrak{c},n}(t,P)\,\mathrm{d}t \qquad (n \in \mathbb{N}), \tag{6.7.23}$$

where, for $n, m \in \mathbb{N}$,

$$F_n(m) = \left|c_{V(m)}^{\mathfrak{d}}\left(\omega; \nu_{V(m)}, p_{V(m)}\right)\right|^2 f_n\left(\nu_{V(m)}, p_{V(m)}\right), \tag{6.7.24}$$

and, for $n \in \mathbb{N}$, $\mathfrak{c} \in \mathfrak{C}(\Gamma)$ and $(t,P) \in \mathbb{R}\times\mathbb{Z}$,

$$\Phi_{\mathfrak{c},n}(t,P) = \left|B_{\mathfrak{c}}^{\mathfrak{d}}\left(\omega; it, \tfrac{1}{2}\,[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]\,P\right)\right|^2 f_n\left(it, \tfrac{1}{2}\,[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]\,P\right) . \tag{6.7.25}$$

By Remark 1.9.2, each term in the sequence $(\nu_{V(1)}, p_{V(1)}), (\nu_{V(1)}, p_{V(1)}), \ldots$ is contained in the set E defined in (6.7.13), and so, by virtue of the hypothesis that condition (ii) of the lemma is satisfied, it follows that the sequence $(F_n)_{n\in\mathbb{N}}$ defined in (6.7.24) satisfies the condition (6.7.9) of Lemma 6.7.4. Consequently, by applying Lemma 6.7.4 and the definition (6.7.24), we find that

$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{m=1}^{\infty} F_n(m)\right) = \sum_{m=1}^{\infty}\left(\lim_{n\to\infty} F_n(m)\right) &= \sum_{m=1}^{\infty}\left|c_{V(m)}^{\mathfrak{d}}\left(\omega; \nu_{V(m)}, p_{V(m)}\right)\right|^2\left(\lim_{n\to\infty} f_n\left(\nu_{V(m)}, p_{V(m)}\right)\right) = \\ &= \sum_{m=1}^{\infty}\left|c_{V(m)}^{\mathfrak{d}}\left(\omega; \nu_{V(m)}, p_{V(m)}\right)\right|^2 f\left(\nu_{V(m)}, p_{V(m)}\right) = \\ &= \sum_{V}\left|C_V^{\mathfrak{d}}\left(\omega; \nu_V, p_V\right)\right|^2 f(\nu_V, p_V)\end{aligned} \tag{6.7.26}$$

(the penultimate equality following by virtue of the hypothesis that the condition (iii) of the lemma is satisfied). Moreover, since one may infer from (6.7.21), (6.7.22) and (6.7.25) that, for $n \in \mathbb{N}$, $\mathfrak{c} \in \mathfrak{C}(\Gamma)$ and $P \in \mathbb{Z}$, the mapping $t \mapsto \Phi_n(\mathfrak{c}; t, P)$ is measurable with respect to the Lebesgue measure on $\mathbb{R}$, one finds (in parallel with the preceding) that, for each $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, the sequence $(\Phi_n)_{n\in\mathbb{N}} = (\Phi_{\mathfrak{c},n})_{n\in\mathbb{N}}$ satisfies the hypotheses of Lemma 6.7.5. Therefore it follows by Lemma 6.7.5 that, for each cusp $\mathfrak{c} \in \mathfrak{C}(\Gamma)$, one has

$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{P\in\mathbb{Z}}\int_{-\infty}^{\infty}\Phi_{\mathfrak{c},n}(t,P)\,\mathrm{d}t\right) &= \sum_{P\in\mathbb{Z}}\int_{-\infty}^{\infty}\left(\lim_{n\to\infty}\Phi_{\mathfrak{c},n}(t,P)\right)\mathrm{d}t = \\ &= \sum_{P\in\mathbb{Z}}\int_{-\infty}^{\infty}\left|B_{\mathfrak{c}}^{\mathfrak{d}}\left(\omega; it, \tfrac{1}{2}\,[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]\,P\right)\right|^2\left(\lim_{n\to\infty} f_n\left(it, \tfrac{1}{2}\,[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]\,P\right)\right)\mathrm{d}t = \\ &= \sum_{p\in\frac{1}{2}[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]\mathbb{Z}}\int_{-\infty}^{\infty}\left|B_{\mathfrak{c}}^{\mathfrak{d}}\left(\omega; it, p\right)\right|^2 f(it,p)\,\mathrm{d}t .\end{aligned} \tag{6.7.27}$$

By Lemma 2.2 the set $\mathfrak{C}(\Gamma)$ is finite. Therefore it follows by (6.7.22), (6.7.23), (6.7.26) and (6.7.27) that one has

$$\begin{aligned}\infty > \lim_{n\to\infty} Y_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f_n) &= \\ &= \sum_{V}\left|C_V^{\mathfrak{d}}\left(\omega; \nu_V, p_V\right)\right|^2 f\left(\nu_V, p_V\right) + \sum_{\mathfrak{c}\in\mathfrak{C}(\Gamma)}\frac{1}{4\pi[\Gamma_{\mathfrak{c}} : \Gamma'_{\mathfrak{c}}]}\sum_{p\in\frac{1}{2}[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]\mathbb{Z}}\int_{-\infty}^{\infty}\left|B_{\mathfrak{c}}^{\mathfrak{d}}\left(\omega; it, p\right)\right|^2 f(it,p)\mathrm{d}t .\end{aligned}$$

This implies that all the sums and integrals occurring on the right-hand side of Equation (6.7.21) are absolutely convergent when $h = f$, and so implies also that $Y_{\omega,\omega}^{\mathfrak{d},\mathfrak{d}}(f)$ is defined, and that the equation

(6.7.21) holds when $h = f$; by reformulating the above result, more concisely, as the statement that one has $\infty > \lim_{n\to\infty} Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f_n) = Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(f)$, one completes the proof of (6.7.16), and so too that of the lemma ■

Lemma 6.7.7. *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of mappings from $\mathbb{N}$ into $\mathbb{C}$ such that, for each $m \in \mathbb{N}$, the sequence $(F_n(m))_{n\in\mathbb{N}}$ is convergent. Suppose moreover that, for some mapping D from $\mathbb{N}$ into $[0,\infty)$, one has both*

$$D(m) \geq |F_n(m)| \qquad (m, n \in \mathbb{N}). \tag{6.7.28}$$

and

$$\sum_{m=1}^{\infty} D(m) < \infty \tag{6.7.29}$$

Then

$$\sum_{m=1}^{\infty} \Big| \lim_{n\to\infty} F_n(m) \Big| < \infty \ , \tag{6.7.30}$$

and, for some $\lambda \in \mathbb{C}$, both equalities in (6.7.10) hold simultaneously.

Proof. See the end of the paragraph above Lemma 6.7.3 ■

Lemma 6.7.8. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of mappings from $\mathbb{R}\times\mathbb{Z}$ into $\mathbb{C}$ such that, for each $(t,p) \in \mathbb{R}\times\mathbb{Z}$, the sequence $(\Phi_n(t,p))_{n\in\mathbb{N}}$ is convergent. Suppose moreover that, for all $n \in \mathbb{N}$ and all $p \in \mathbb{Z}$, the mapping $t \mapsto \Phi_n(t,p)$ is a complex valued function on $\mathbb{R}$ that is measurable with respect to the Lebesgue measure on $\mathbb{R}$, and that, for some mapping Δ from $\mathbb{R}\times\mathbb{Z}$ into $[0,\infty)$, one has both*

$$\Delta(t,p) \geq |\Phi_n(t,p)| \qquad (t \in \mathbb{R},\ p \in \mathbb{Z},\ n \in \mathbb{N}) \tag{6.7.31}$$

and

$$\sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta(t,p)\, \mathrm{d}t < \infty \ . \tag{6.7.32}$$

Then

$$\sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \Big| \lim_{n\to\infty} \Phi_n(t,p) \Big| \,\mathrm{d}t < \infty \tag{6.7.33}$$

and, for some $\lambda \in \mathbb{C}$, both equalities in (6.7.12) hold simultaneously.

Proof. See the end of the paragraph above Lemma 6.7.3 ■

Lemma 6.7.9. *Let $\sigma \in (1/2,1)\cup(1,2)$ and $\varrho, \vartheta \in (3,\infty)$. Let $d \in \mathcal{H}^{\sigma}_{\star}(\varrho,\vartheta)$ be such that the sum formulae for $Y^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_1}(d)$ and $Y^{\mathfrak{b},\mathfrak{b}}_{\omega_2,\omega_2}(d)$ are valid. Suppose moreover that E is the set defined in (6.7.13), that $f \in \mathcal{H}^{\sigma}_{\star}(\varrho,\vartheta)$, and that $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{H}^{\sigma}_{\star}(\varrho,\vartheta)$ which satisfies the condition (6.7.6) of Lemma 6.7.3 and the following conditions:*

- *(i) for all $n \in \mathbb{N}$, the sum formula for $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f_n)$ is valid;*
- *(ii) for all $(\nu,p) \in E$ and all $n \in \mathbb{N}$, one has $d(\nu,p) \geq |f_n(\nu,p)|$;*
- *(iii) for all $(\nu,p) \in \mathbb{C}\times\mathbb{Z}$ such that $|\mathrm{Re}(\nu)| \leq \sigma$, one has $\lim_{n\to\infty} f_n(\nu,p) = f(\nu,p)$.*

Then the sum formula for $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f)$ is valid, and one has

$$\chi^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f) = \lim_{n\to\infty} \chi^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f_n) \ , \tag{6.7.34}$$

$$X^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f) = \lim_{n\to\infty} X^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f_n) \ , \tag{6.7.35}$$

$$Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f) = \lim_{n\to\infty} Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f_n) \ . \tag{6.7.36}$$

Proof. Since $\mathcal{H}^\sigma_\star(\varrho,\vartheta)\supset\{f_n : n\in\mathbb{N}\}\cup\{f,d\}$, it follows from the definition (6.7.1) that one has

$$\mathbb{C}\ni\chi^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(h)=\frac{1}{4\pi^3}\,\delta^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}\sum_{p\in\mathbb{Z}}\int_{-\infty}^{\infty}h(it,p)\left(t^2+p^2\right)\mathrm{d}t\qquad\text{for all}\quad h\in\{f_n : n\in\mathbb{N}\}\cup\{f,d\}. \tag{6.7.37}$$

In particular, since the condition (ii) of the lemma implies that $d(\nu,p)\geq 0$ for all $(\nu,p)\in E$, one has

$$0\leq\sum_{p\in\mathbb{Z}}\int_{-\infty}^{\infty}d(it,p)\left(t^2+p^2\right)\mathrm{d}t<\infty\ . \tag{6.7.38}$$

By virtue of (6.7.13), (6.7.37), (6.7.38) and the conditions (ii) and (iii) of the lemma, one may verify that the hypotheses of Lemma 6.7.8 are satisfied when, for each $n\in\mathbb{N}$, the mapping $\Phi_n:\mathbb{R}\times\mathbb{Z}\to\mathbb{C}$ is given by $\Phi_n(t,p)=(t^2+p^2)f_n(it,p)$ (note that the conditions (6.7.31) and (6.7.32) are then satisfied if one specifies that $\Delta(t,p)=(t^2+p^2)d(it,p)$ for $t\in\mathbb{R}$, $p\in\mathbb{Z}$). Therefore, bearing in mind the condition (iii) of the lemma, it follows by Lemma 6.7.8 and (6.7.37) that the equation (6.7.34) holds.

We observe next that, since f and the sequence $(f_n)_{n\in\mathbb{N}}$ satisfy the relevant hypotheses of Lemma 6.7.3, the result (6.7.35) may therefore be obtained similarly to how (6.7.15) was obtained (within the proof of Lemma 6.7.6): note, with regard to the relevant application of Lemma 6.5.9, that no distinction need be made between those cases where $\mathfrak{a}=\mathfrak{b}$ and $\omega_1=\omega_2$ and those where either $\mathfrak{a}\neq\mathfrak{b}$ or else $\omega_1\neq\omega_2$.

Given that the proofs of the the results (6.7.34) and (6.7.35) have been adequately described, and given our hypothesis that the condition (i) of the lemma is satisfied, it will suffice for the completion of the proof of the lemma that we show now that the equation (6.7.36) holds (our reasoning on this point is similar to that used in the paragraph containing (6.7.19), within our proof of Lemma 6.7.6).

We begin the proof of (6.7.36) by observing that it follows from the condition (i) of the lemma and the definition (6.7.3) that, for all $n\in\mathbb{N}$, one has

$$Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f_n)=\sum_{m=1}^{\infty}F_n(m)+\sum_{\mathfrak{c}\in\mathfrak{C}(\Gamma)}\frac{1}{4\pi\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]}\sum_{P\in\mathbb{Z}}\int_{-\infty}^{\infty}\Phi_{\mathfrak{c},n}(t,P)\,\mathrm{d}t\ , \tag{6.7.39}$$

where

$$\Phi_{\mathfrak{c},n}(t,P)=\overline{B^{\mathfrak{a}}_{\mathfrak{c}}\left(\omega_1;it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\right)}\,B^{\mathfrak{b}}_{\mathfrak{c}}\left(\omega_2;it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\right)f_n\big(it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\big) \tag{6.7.40}$$

and

$$F_n(m)=\overline{c^{\mathfrak{a}}_{V(m)}\left(\omega_1;\nu_{V(m)},p_{V(m)}\right)}\,c^{\mathfrak{b}}_{V(m)}\left(\omega_2;\nu_{V(m)},p_{V(m)}\right)f_n\big(\nu_{V(m)},p_{V(m)}\big)\ , \tag{6.7.41}$$

with the sequence of spaces $V(1),V(2),\ldots$ being as indicated between (6.7.22) and (6.7.23), within the proof of Lemma 6.7.6. Note that, by (6.7.40), (6.7.41), the condition (iii) of the lemma and the hypothesis that $f\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$, one has, for $m\in\mathbb{N}$,

$$\lim_{n\to\infty}F_n(m)=\overline{c^{\mathfrak{a}}_{V(m)}\big(\omega_1;\nu_{V(m)},p_{V(m)}\big)}\,c^{\mathfrak{b}}_{V(m)}\big(\omega_2;\nu_{V(m)},p_{V(m)}\big)\,f\big(\nu_{V(m)},p_{V(m)}\big)\in\mathbb{C} \tag{6.7.42}$$

and, for $(t,P)\in\mathbb{R}\times\mathbb{Z}$ and $\mathfrak{c}\in\mathfrak{C}(\Gamma)$,

$$\lim_{n\to\infty}\Phi_{\mathfrak{c},n}(t,P)=\overline{B^{\mathfrak{a}}_{\mathfrak{c}}\left(\omega_1;it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\right)}\,B^{\mathfrak{b}}_{\mathfrak{c}}\left(\omega_2;it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\right)f\big(it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\big)\in\mathbb{C}\ . \tag{6.7.43}$$

Since the sum formulae for $Y^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_1}(d)$ and $Y^{\mathfrak{b},\mathfrak{b}}_{\omega_2,\omega_2}(d)$ are valid, one has also

$$\mathbb{C}\ni\frac{Y^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_1}(d)+Y^{\mathfrak{b},\mathfrak{b}}_{\omega_2,\omega_2}(d)}{2}=\sum_{m=1}^{\infty}D(m)+\sum_{\mathfrak{c}\in\mathfrak{C}(\Gamma)}\frac{1}{4\pi\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]}\sum_{P\in\mathbb{Z}}\int_{-\infty}^{\infty}\Delta_{\mathfrak{c}}(t,P)\,\mathrm{d}t\ , \tag{6.7.44}$$

where

$$D(m) = \frac{1}{2}\left(\left|c^{\mathfrak{a}}_{V(m)}\big(\omega_1;\nu_{V(m)},p_{V(m)}\big)\right|^2 + \left|c^{\mathfrak{b}}_{V(m)}\big(\omega_2;\nu_{V(m)},p_{V(m)}\big)\right|^2\right) d\left(\nu_{V(m)},p_{V(m)}\right) \tag{6.7.45}$$

and

$$\Delta_{\mathfrak{c}}(t,P) = \frac{1}{2}\left(\left|B^{\mathfrak{a}}_{\mathfrak{c}}\left(\omega_1;it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\right)\right|^2 + \left|B^{\mathfrak{b}}_{\mathfrak{c}}\left(\omega_2;it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\right)\right|^2\right) d\big(it,\tfrac{1}{2}\left[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}\right]P\big)\ . \tag{6.7.46}$$

Note that, by the condition (ii) of the lemma, and the definition (6.7.13), all terms of the sums occurring on the right-hand side of the equality sign in (6.7.44) are real and non-negative; it therefore follows from (6.7.44) that the function $D : \mathbb{N} \to [0,\infty)$ satisfies the condition (6.7.29) of Lemma 6.7.7, and that one has

$$\sum_{P=-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta_{\mathfrak{c}}(t,P)\,\mathrm{d}t < \infty \qquad \text{for } \mathfrak{c} \in \mathfrak{C}(\Gamma).$$

Moreover, by (6.7.40), (6.7.41), (6.7.45), (6.7.46), the arithmetic-geometric mean inequality, Remark 1.9.2 and the condition (ii) of the lemma, one finds that D and the sequence $(F_n)_{n\in\mathbb{N}}$ satisfy the condition (6.7.28) of Lemma 6.7.7, and that

$$\Delta_{\mathfrak{c}}(t,p) \geq |\Phi_{\mathfrak{c},n}(t,P)| \qquad\qquad (\mathfrak{c} \in \mathfrak{C}(\Gamma),\ (t,P) \in \mathbb{R}\times\mathbb{Z},\ n \in \mathbb{N}).$$

Given (6.7.42), (6.7.43) and the points just noted in the preceding paragraph, and given that (6.7.39) holds for all $n \in \mathbb{N}$, it follows by Lemma 6.7.7 and Lemma 6.7.8 that one has

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} F_n(m) = \sum_{m=1}^{\infty} \overline{c^{\mathfrak{a}}_{V(m)}\left(\omega_1;\nu_{V(m)},p_{V(m)}\right)}\, c^{\mathfrak{b}}_{V(m)}\left(\omega_2;\nu_{V(m)},p_{V(m)}\right) f\big(\nu_{V(m)},p_{V(m)}\big) \tag{6.7.47}$$

and, for each $\mathfrak{c} \in \mathfrak{C}(\Gamma)$,

$$\lim_{n\to\infty} \sum_{P\in\mathbb{Z}} \int_{-\infty}^{\infty} \Phi_{\mathfrak{c},n}(t,P)\,\mathrm{d}t = \frac{1}{i} \sum_{p\in\frac{1}{2}[\Gamma_{\mathfrak{c}}:\Gamma'_{\mathfrak{c}}]\mathbb{Z}} \int_{(0)} \overline{B^{\mathfrak{a}}_{\mathfrak{c}}\left(\omega_1;\nu,p\right)}\, B^{\mathfrak{b}}_{\mathfrak{c}}\left(\omega_2;\nu,p\right) f(\nu,p)\,\mathrm{d}\nu\ . \tag{6.7.48}$$

It is worth clarifying here that, by virtue of the results (6.7.30) and (6.7.33) of the lemmas just applied, all of the integrals and sums occurring on the right-hand sides of the equations (6.7.47) and (6.7.48) are absolutely convergent. Therefore, since the set $\mathfrak{C}(\Gamma)$ is finite, and since $\{V(m) : m \in \mathbb{N}\}$ is the set of cuspidal irreducible subspaces occurring in the decomposition (1.7.4) of the space ${}^0L^2(\Gamma\backslash G)$, it follows that all of the integrals and sums occurring on the right-hand side of Equation (6.7.3) are absolutely convergent when $\mathfrak{d} = \mathfrak{a}$, $\mathfrak{d}' = \mathfrak{b}$, $\omega = \omega_1$, $\omega' = \omega_2$ and $h = f$. The case $(\mathfrak{d},\mathfrak{d}',\omega,\omega',h) = (\mathfrak{a},\mathfrak{b},\omega_1,\omega_2,f)$ of (6.7.3) therefore defines $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f)$, and defines it in such a way that one has $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f) \in \mathbb{C}$. We observe moreover that the limits occurring on the left-hand sides of the equations (6.7.47) and (6.7.48) exist (i.e. they are limits of convergent sequences), and so, by (6.7.39) (for $n \in \mathbb{N}$), the case $(\mathfrak{d},\mathfrak{d}',\omega,\omega',h) = (\mathfrak{a},\mathfrak{b},\omega_1,\omega_2,f)$ of (6.7.3) and the finiteness of the set $\mathfrak{C}(\Gamma)$, it follows from (6.7.47) and (6.7.48) that the equation (6.7.36) holds ∎

Lemma 6.7.10. *Let $\sigma \in (1,2)$, let $\varrho,\vartheta \in (3,\infty)$, and let $f \in \mathcal{H}^{\sigma}_{\star}(\varrho,\vartheta)$. Put*

$$C_f(\varrho,\vartheta) = \sup\left\{(1+|\mathrm{Im}(\nu)|)^{\varrho}\,(1+|p|)^{\vartheta}\,|f(\nu,p)| \,:\, p\in\mathbb{Z},\ \nu\in\mathbb{C} \text{ and } |\mathrm{Re}(\nu)| \leq \sigma\right\}\ . \tag{6.7.49}$$

Suppose moreover that $\ell \in \mathbb{N}$, and that f_ℓ is the mapping from $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{Z}$ into $\mathbb{C}$ given by

$$f_\ell(\nu,p) = \begin{cases} f(\nu,p)\exp\left(-\nu^4/\ell\right) & \text{if } |p| \leq \ell; \\ 0 & \text{otherwise.}\end{cases} \tag{6.7.50}$$

Then $f_\ell \in \mathcal{H}^\sigma_\star(\varrho,\vartheta)$, one has

$$\sup\Big\{(1+|\mathrm{Im}(\nu)|)^\varrho\,(1+|p|)^\vartheta\,|f_\ell(\nu,p)|\,:\,p\in\mathbb{Z},\ \nu\in\mathbb{C}\text{ and }|\mathrm{Re}(\nu)|\le\sigma\Big\}\le\exp\big(8\sigma^4/\ell\big)\,C_f(\varrho,\vartheta)<\infty\ ,\tag{6.7.51}$$

and, for $\mathfrak{d},\mathfrak{d}'\in\{\mathfrak{a},\mathfrak{b}\}$ and $\omega,\omega'\in\{\omega_1,\omega_2\}$, the sum formula for $Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(f_\ell)$ is valid.

Proof. Since $f\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$, since the function $p\mapsto|p|$ is even, and since the function $\nu\mapsto\exp(-\nu^4/\ell)$ is both even and holomorphic on $\mathbb{C}$, one finds, when checking that $f_\ell\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$, that it is only the condition (iii) of Theorem B that requires more than cursory consideration; moreover, given the definition (6.7.50), and given that $f\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$ and $\ell\in\mathbb{N}$, one need do little more than observe that

$$\big|\exp\left(-\nu^4/\ell\right)\big|\le\exp\left(8\sigma^4/\ell\right)\qquad\qquad(|\mathrm{Re}(\nu)|\le\sigma)\tag{6.7.52}$$

in order to verify that the condition (iii) of Theorem B is satisfied when $h=f_\ell$. Since further discussion of the proof that f_ℓ lies in the space $\mathcal{H}^\sigma_\star(\varrho,\vartheta)$ is probably unnecessary, we skip it.

We can also be brief in discussing the proof of the result (6.7.51): this result follows, by virtue of (6.7.52), from the definitions (6.7.49) and (6.7.50), and the hypothesis that $f\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$.

What remains to be demonstrated is the validity of the sum formula for $Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}$, when $\mathfrak{d},\mathfrak{d}'\in\{\mathfrak{a},\mathfrak{b}\}$ and $\omega,\omega'\in\{\omega_1,\omega_2\}$. Let $\mathfrak{d},\mathfrak{d}'\in\{\mathfrak{a},\mathfrak{b}\}$ and $\omega,\omega'\in\{\omega_1,\omega_2\}$. Suppose also that λ^*_ℓ is the mapping from $\mathbb{C}\times\{-\ell,1-\ell,\ldots,\ell\}$ into $\mathbb{C}$ defined by the equations (6.6.4) of Proposition 6.6.1. As a first step towards verifying the sum formula for $Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}$, we observe that, when $p\in\mathbb{Z}$, $\nu\in\mathbb{C}$ and $|\mathrm{Re}(\nu)|\le\sigma$, it follows from (6.7.50) that

$$f_\ell(\nu,p)=\begin{cases}\lambda^*_\ell(\nu,p)\,\overline{\theta_\ell(-\overline{\nu}\,,p)}\,\eta_\ell(\nu,p) & \text{if }|p|\le\ell,\\ 0 & \text{otherwise,}\end{cases}\tag{6.7.53}$$

where η_ℓ and θ_ℓ are the mappings from $\{\nu\in\mathbb{C}:|\mathrm{Re}(\nu)|\le\sigma\}\times\{-\ell,1-\ell,\ldots,\ell\}$ into $\mathbb{C}$ given by

$$\eta_\ell(\nu,p)=\frac{f(\nu,p)}{\lambda^*_\ell(\nu,p)}\ ,\tag{6.7.54}$$

$$\theta_\ell(\nu,p)=\exp\left(-\nu^4/\ell\right)\ .\tag{6.7.55}$$

In cases where $\lambda^*_\ell(\nu,p)=0$, we take (6.7.54) to mean that $\eta_\ell(\nu,p)=\lim_{\delta\to0+}f(\nu+i\delta,p)/\lambda^*_\ell(\nu+i\delta,p)$.

If the functions η_ℓ and θ_ℓ lie in the space $\mathcal{T}^\ell_\sigma$ defined in the paragraph containing (6.4.3), then by Proposition 6.6.1 (applied with η_ℓ, θ_ℓ, $\mathfrak{d}$, $\mathfrak{d}'$, ω and ω' substituted for η, θ, $\mathfrak{a}$, $\mathfrak{b}$, ω_1 and ω_2, respectively) one obtains, given (6.7.53), the case $h=f_\ell$ of Equation (6.7.4) (the terms occurring in that equation being defined by (6.7.1)-(6.7.3)). Moreover, by the same application of Proposition 6.6.1, one finds that if $\eta_\ell,\theta_\ell\in\mathcal{T}^\ell_\sigma$ then the sums and integrals involved in the definition (via (6.7.3)) of $Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(f_\ell)$ are all absolutely convergent. One may therefore conclude that

$$\text{the sum formula for }Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(f_\ell)\ \text{ is valid if }\ \eta_\ell,\theta_\ell\in\mathcal{T}^\ell_\sigma.\tag{6.7.56}$$

We show next that if $\eta\in\{\eta_\ell,\theta_\ell\}$ then the conditions (T1)-(T3) below (6.4.3) are satisfied; we thereby establish that the functions η_ℓ and θ_ℓ do lie in the space $\mathcal{T}^\ell_\sigma$.

Starting with the easier case, $\eta=\theta_\ell$, we note firstly that the mapping $(\nu,p)\mapsto(-\nu,-p)$ is a permutation of the domain of θ_ℓ, and that by (6.7.55) one has $\theta_\ell(-\nu,-p)=\exp(-(-\nu)^4/\ell)=\exp(-\nu^4/\ell)=\theta_\ell(\nu,p)$, for all (ν,p) in that domain. Therefore the condition (T1) below (6.4.3) is satisfied when $\eta=\theta_\ell$. Since the complex function $\nu\mapsto\exp(-\nu^4/\ell)$ is entire, we find, secondly, that the condition (T2) is satisfied when $\eta=\theta_\ell$. Thirdly, we note that, for $A>0$, $\alpha\in[-\sigma,-\sigma]$, $t\ge0$ and $\nu=\alpha\pm it$, one has

$$\begin{aligned}\frac{\big|\exp\left(-\nu^4/\ell\right)\big|}{(1+|\mathrm{Im}(\nu)|)^{-A}\,e^{-(\pi/2)|\mathrm{Im}(\nu)|}}&\le\exp\left(\left(A+\frac{\pi}{2}\right)t-\frac{\left(t^4+\alpha^4-6\alpha^2t^2\right)}{\ell}\right)\le\\&\le\exp\left(\max_{x\in\mathbb{R}}\left((A+\frac{\pi}{2})\,\ell^{1/4}x+6\sigma^2\ell^{-1/2}x^2-x^4\right)\right)<\infty\ ,\end{aligned}$$

so that the condition (T3) is satisfied when $\eta = \theta_\ell$.

We consider next the conditions (T1)-(T3) in the case where η is equal to the function η_ℓ defined in Equation (6.7.54).

Given the definition of $\lambda_\ell^*(\nu,p)$ in (6.6.4) (noting, in particular, the second equality there), and given that ℓ is a positive integer, it follows that for each $p \in \{-\ell, 1-\ell, \ldots, \ell\}$ the mapping $\nu \mapsto (\nu^2-1)^2/\lambda_\ell^*(\nu,p)$ is is a holomorphic function on the open strip $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| < 2\}$. Therefore, since the hypotheses that $\sigma \in (1,2)$ and $f \in \mathcal{H}_\star^\sigma(\varrho,\vartheta)$ imply that for each $p \in \mathbb{Z}$ the mapping $\nu \mapsto f(\nu,p)/(\nu^2-1)^2$ can be holomorphically continued into a neighbourhood of the closed strip $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\}$, it follows from (6.7.54) that the condition (T2) below (6.4.3) is satisfied when $\eta = \eta_\ell$.

By the definition (6.6.4) and the hypothesis that $f \in \mathcal{H}_\star^\sigma(\varrho,\vartheta)$, one has both $1/\lambda_\ell^*(-\nu,-p) = 1/\lambda_\ell^*(\nu,p)$ and $f(-\nu,-p) = f(\nu,p)$ for all $(\nu,p) \in (\mathbb{C}-\mathbb{Z}) \times \{-\ell, 1-\ell, \ldots, \ell\}$ such that $|\mathrm{Re}(\nu) \leq \sigma$. Therefore, given the definition (6.7.54), and given that the condition (T2) below (6.4.3) is satisfied when $\eta = \eta_\ell$, one may deduce that $\eta_\ell(-\nu,-p) = \eta_\ell(\nu,p)$ for all $(\nu,p) \in \mathbb{C} \times \mathbb{Z}$ such that $|\mathrm{Re}(\nu)| \leq \sigma$ and $|p| \leq \ell$. It follows that the condition (T1) below (6.4.3) is satisfied when $\eta = \eta_\ell$.

We now have only to check that the condition (T3) below (6.4.3) is satisfied when $\eta = \eta_\ell$; in order that this be verified, it will suffice to show that, for all $A > 0$, one has

$$\sup\left\{(1+|\mathrm{Im}(\nu)|)^A e^{(\pi/2)|\mathrm{Im}(\nu)|} |\eta_\ell(\nu,p)| \ : \ \nu \in \mathbb{C},\ p \in \mathbb{Z},\ |\mathrm{Re}(\nu)| \leq \sigma \text{ and } |p| \leq \ell\right\} < \infty . \qquad (6.7.57)$$

Since the condition (T2) below (6.4.3) is satisfied when $\eta = \eta_\ell$, the function $(\nu,p) \mapsto |\eta_\ell(\nu,p)|$ is bounded on the compact set $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)|, |\mathrm{Im}(\nu)| \leq \sigma\} \times \{p \in \mathbb{Z} : |p| \leq \ell\} = T(\sigma,\ell)$ (say). Moreover, given any $A > 0$, the function $(\nu,p) \mapsto (1+|\mathrm{Im}(\nu)|)^A e^{(\pi/2)|\mathrm{Im}(\nu)|}$ is bounded on the same set, $T(\sigma,\ell)$. Therefore the condition (6.7.57) is satisfied if and only if it is the case that, for each $A > 0$, the function $(\nu,p) \mapsto (1+|\mathrm{Im}(\nu)|)^A e^{(\pi/2)|\mathrm{Im}(\nu)|}|\eta_\ell(\nu,p)|$ is bounded on the set $U(\sigma,\ell)$ given by

$$U(\sigma,\ell) = \{(\nu,p) \in \mathbb{C} \times \mathbb{Z} : \ |\mathrm{Re}(\nu)| \leq \sigma \leq |\mathrm{Im}(\nu)| \ \text{ and } \ |p| \leq \ell\} . \qquad (6.7.58)$$

Since $\sigma \in (1,2)$, and since $\ell \in \mathbb{N}$, it follows by (6.6.4), (6.4.5) and (6.7.58) that

$$\begin{aligned}\frac{1}{|\lambda_\ell^*(\nu,p)|} &= \frac{\pi|\nu|}{|\sin(\pi\nu)|}\,\frac{|(\nu+p)(\nu-p)|}{|\nu|^{1+\epsilon(p)}} \prod_{\substack{0<m\leq\ell\\ m\neq|p|}} \frac{1}{|(\nu+m)(\nu-m)|} \leq \\ &\leq \frac{2\pi\,(1+|\mathrm{Im}(\nu)|)}{\sinh(\pi\,|\mathrm{Im}(\nu)|)}\,\frac{4\ell^2\,(1+|\mathrm{Im}(\nu)|)^2}{|\mathrm{Im}(\nu)|^{2\ell+2\epsilon(p)}} \leq \frac{8\pi\ell^2\,(1+|\mathrm{Im}(\nu)|)^3}{(1/3)\exp(\pi\,|\mathrm{Im}(\nu)|)} \qquad \text{for } (\nu,p) \in U(\sigma,\ell). \qquad (6.7.59)\end{aligned}$$

We are also given that $f \in \mathcal{H}_\star^\sigma(\varrho,\vartheta)$, and that $(\varrho,\vartheta) \in (3,\infty)$, and so may deduce from (6.7.54), (6.7.49), (6.7.58), (6.7.59) and the final inequality in (6.7.51) that, for all real $A > 0$, one has

$$\begin{aligned}&\sup_{(\nu,p)\in U(\sigma,\ell)} (1+|\mathrm{Im}(\nu)|)^A e^{(\pi/2)|\mathrm{Im}(\nu)|} |\eta_\ell(\nu,p)| \leq \\ &\qquad \leq 24\pi\ell^2 C_f(\varrho,\vartheta) \sup_{(\nu,p)\in U(\sigma,\ell)} (1+|\mathrm{Im}(\nu)|)^{A+3-\varrho} (1+|p|)^{-\vartheta} e^{-(\pi/2)|\mathrm{Im}(\nu)|} \leq \\ &\qquad \leq 24\pi\ell^2 C_f(\varrho,\vartheta) \max_{t\geq 0} (1+t)^A e^{-(\pi/2)t} < \infty .\end{aligned}$$

By this result, in combination with what has been discussed in the paragraph containing (6.7.58), it follows that the condition (6.7.57) is satisfied. It has therefore been verified that the condition (T3) below (6.4.3) is satisfied when $\eta = \eta_\ell$.

We have now found that each of the conditions (T1)-(T3) below (6.4.3) is satisfied if one has either $\eta = \theta_\ell$ or $\eta = \eta_\ell$. Therefore the functions θ_ℓ and η_ℓ lie in the space $\mathcal{T}_\sigma^\ell$, and so, by (6.7.56), the sum formula for $Y_{\omega,\omega'}^{\mathfrak{d},\mathfrak{d}'}(f_\ell)$ is valid ■

Lemma 6.7.11. *Let $\sigma \in (1,2)$, let $\varrho,\vartheta \in (3,\infty)$, and let E be the set defined in (6.7.13). Suppose moreover that $d^\flat$ is the mapping from $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{Z}$ into $\mathbb{C}$ given by*

$$d^\flat(\nu,p) = \left(1-\nu^2\right)^2 \left(4-\nu^2\right)^{-(\varrho+4)/2} (1+|p|)^{-\vartheta} . \qquad (6.7.60)$$

Then $d^\flat \in \mathcal{H}^\sigma_\star(\varrho,\vartheta)$, one has

$$d^\flat(\nu,p) \geq 2^{-(\varrho+5)}\left(1+|\mathrm{Im}(\nu)|\right)^{-\varrho}\left(1+|p|\right)^{-\vartheta} \qquad ((\nu,p)\in E), \tag{6.7.61}$$

and, for $\mathfrak{d}\in\{\mathfrak{a},\mathfrak{b}\}$ and $\omega\in\{\omega_1,\omega_2\}$, the sum formula for $Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(d^\flat)$ is valid.

Proof. The function $d^\flat$ defined in (6.7.60) is a very minor modification of the function '$f_{a,b}$' which is defined and used in the proof of Proposition 11.3.2 of [32]. Since one has $\mathrm{Re}(4-\nu^2)\geq 4-\sigma^2>0$ when $|\mathrm{Re}(\nu)|\leq\sigma$, it follows that the mapping $\nu\mapsto(1-\nu^2)^2(4-\nu^2)^{-(\varrho+4)/2}$ is holomorphic on a neighbourhood of the strip $\{\nu\in\mathbb{C}:|\mathrm{Re}(\nu)|\leq\sigma\}$ (i.e. one has $(4-\nu^2)^{-(\varrho+4)/2}=\exp(-(1/2)(\varrho+4)\log(4-\nu^2))$, where $\log(z)$ denotes the principal branch of the logarithm function). Consequently the condition (ii) of Theorem B is satisfied when $h=d^\flat$. Given (6.7.60), and given that $1<\sigma<2$, one has also $d^\flat(-\nu,-p)=d^\flat(\nu,p)$ and, via a short calculation, $|d^\flat(\nu,p)|\leq 2^{(\varrho+4)/2}(\sigma+1)^4(2-\sigma)^{-(\varrho+4)}(1+|\mathrm{Im}(\nu)|)^{-\varrho}(1+|p|)^{-\vartheta}$ for all $p\in\mathbb{Z}$ and all $\nu\in\mathbb{C}$ such that $|\mathrm{Re}(\nu)|\leq\sigma$. Therefore the conditions (i) and (iii) of Theorem B are satisfied when $h=d^\flat$. Since it is moreover the case that the function $\nu\mapsto(1-\nu^2)^2(4-\nu^2)^{-(\varrho+4)/2}$ has a zero of order 2 at $\nu=1$, we may conclude (given what has already been noted above) that $d^\flat$ lies in the space $\mathcal{H}^\sigma_\star(\varrho,\vartheta)$.

When $(\nu,p)\in(i\mathbb{R})\times\mathbb{Z}$, the inequality stated in (6.7.61) follows by virtue of the definition (6.7.60), the hypothesis that one has $\varrho,\vartheta\in(3,\infty)$, and the fact that $0<4+t^2\leq 4(1+t^2)\leq 4(1+|t|)^2$ for $t\in\mathbb{R}$. Consequently, given the definition (6.7.13) of the set E, the proof of (6.7.61) may be completed by making use of the observation that for $\nu\in[-2/9,2/9]$ and $\lambda=1-\nu^2$ one has $1\geq\lambda>3/4$, and so $\lambda^2(\lambda+3)^{-(\varrho+4)/2}\geq 4^{-\varrho/2}(1+3/\lambda)^{-2}\geq 2^{-\varrho}/25>2^{-(\varrho+5)}$.

Suppose now that $\mathfrak{d}\in\{\mathfrak{a},\mathfrak{b}\}$ and $\omega\in\{\omega_1,\omega_2\}$. The proof of the lemma will be complete if we are able to show that the sum formula for $Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(d^\flat)$ is valid. As a first step towards this, we let $d^\flat_\ell$ denote (when $\ell\in\mathbb{N}$) the mapping from $\{\nu\in\mathbb{C}:|\mathrm{Re}(\nu)|\leq\sigma\}\times\mathbb{Z}$ into $\mathbb{C}$ given by

$$d^\flat_\ell(\nu,p)=\begin{cases} d^\flat(\nu,p)\exp\left(-\nu^4/\ell\right) & \text{if } |p|\leq\ell;\\ 0 & \text{otherwise.}\end{cases} \tag{6.7.62}$$

By the case $f=d^\flat$ of Lemma 6.7.10, it follows that the condition (6.7.6) of Lemma 6.7.3 is satisfied when $(f_n)_{n\in\mathbb{N}}$ is the sequence $(d^\flat_\ell)_{\ell\in\mathbb{N}}$, and that, for each $\ell\in\mathbb{N}$, the function $d^\flat_\ell$ lies in the space $\mathcal{H}^\sigma_\star(\varrho,\vartheta)$ and the sum formula for $Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(d^\flat_\ell)$ is valid. By (6.7.61), (6.7.62) and (6.7.13), we have moreover

$$0\leq d^\flat_1(\nu,p)\leq d^\flat_2(\nu,p)\leq\ldots \qquad ((\nu,p)\in E)$$

and

$$\lim_{\ell\to\infty} d^\flat_\ell(\nu,p)=d^\flat(\nu,p) \qquad (p\in\mathbb{Z},\ \nu\in\mathbb{C}\text{ and }|\mathrm{Re}(\nu)|\leq\sigma),$$

and so may conclude that, when σ, ϱ, ϑ, $\mathfrak{d}$, ω and E are as we currently suppose, when $f=d^\flat$, and when $(f_n)_{n\in\mathbb{N}}$ is the sequence $(d^\flat_\ell)_{\ell\in\mathbb{N}}$, it is then the case that all of the hypotheses of Lemma 6.7.6 are satisfied. Therefore it follows by Lemma 6.7.6 that the sum formula for $Y^{\mathfrak{d},\mathfrak{d}}_{\omega,\omega}(d^\flat)$ is valid ■

The proof of Proposition 6.7.2. Let σ, ϱ, ϑ and f satisfy the hypotheses of Proposition 6.7.2. For each $\ell\in\mathbb{N}$, define f_ℓ to be the mapping from $\{\nu\in\mathbb{C}:|\mathrm{Re}(\nu)|\leq\sigma\}\times\mathbb{Z}$ into $\mathbb{C}$ given by the equation (6.7.50) of Lemma 6.7.10. Define the set E as in (6.7.13), and put

$$d=D_f(\varrho,\vartheta)d^\flat\ , \tag{6.7.63}$$

where $d^\flat$ is the mapping from $\{\nu\in\mathbb{C}:|\mathrm{Re}(\nu)|\leq\sigma\}\times\mathbb{Z}$ into $\mathbb{C}$ defined in Lemma 6.7.11, while

$$D_f(\varrho,\vartheta)=2^{\varrho+5}\exp\left(8\sigma^4\right)C_f(\varrho,\vartheta) \tag{6.7.64}$$

with $C_f(\varrho,\vartheta)$ being the constant defined in Lemma 6.7.10 (so that, as follows by the result (6.7.51) of that lemma, one has $0\leq C_f(\varrho,\vartheta)<\infty$).

We show next that when σ, ϱ, ϑ and f are as we currently suppose, and when the set E, the function d and the sequence $(f_\ell)_{\ell\in\mathbb{N}}$ are as just defined above, all of the hypotheses of Lemma 6.7.9 are satisfied. Once this is achieved the proof of Proposition 6.7.2 will be essentially complete: for it will then follow, by Lemma 6.7.9, that the sum formula for $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f)$ is valid.

Our current hypotheses concerning σ, ϱ, ϑ and f imply that those of the hypotheses of Lemma 6.7.9 that are concerned solely with σ, ϱ, ϑ and f are satisfied. Likewise, our definition of the set E is the same as that which is posited in Lemma 6.7.9. Consequently the only hypotheses of Lemma 6.7.9 requiring further consideration are those concerning either the function d or the sequence $(f_n)_{n\in\mathbb{N}}$.

By Lemma 6.7.10 it follows that $\mathcal{H}^\sigma_\star(\varrho,\vartheta)\supseteq\{f_n : n\in\mathbb{N}\}$, and that for each $n\in\mathbb{N}$ the sum formula for $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f_n)$ is valid. Given the result (6.7.51) of Lemma 6.7.10, and the definition (6.7.50), it moreover follows that the sequence $(f_n)_{n\in\mathbb{N}}$ satisfies the condition (6.7.6) of Lemma 6.7.3, and that, for all $p\in\mathbb{Z}$, and all $\nu\in\mathbb{C}$ such that $|\mathrm{Re}(\nu)|\leq\sigma$, one has $\lim_{n\to\infty} f_n(\nu,p)=f(\nu,p)$. These observations enable one to conclude that those of the hypotheses of Lemma 6.7.9 that concern $(f_n)_{n\in\mathbb{N}}$, but not d, are satisfied. Therefore it now only remains to be shown that the function d defined in (6.7.63) and (6.7.64) does satisfy the relevant hypotheses of Lemma 6.7.9.

Consider firstly the hypothesis that $d\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$. By Lemma 6.7.11 one has $d^\flat\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$; it therefore follows, given (6.7.63) and (6.7.64), that since $\mathcal{H}^\sigma_\star(\varrho,\vartheta)$ is a vector space over $\mathbb{C}$ one has also $d\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$. Similarly, since Lemma 6.7.11 implies that the sum formulae for $Y^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_1}(d^\flat)$ and $Y^{\mathfrak{b},\mathfrak{b}}_{\omega_2,\omega_2}(d^\flat)$ are valid, one may deduce that the sum formulae for $Y^{\mathfrak{a},\mathfrak{a}}_{\omega_1,\omega_1}(d)$ and $Y^{\mathfrak{b},\mathfrak{b}}_{\omega_2,\omega_2}(d)$ are also valid (this following in view of the intrinsic linearity, as regards their dependence on the test function h, of all the transforms, integrals and sums occurring in the definitions (6.7.1)-(6.7.3) of $\chi^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$, $X^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$ and $Y^{\mathfrak{d},\mathfrak{d}'}_{\omega,\omega'}(h)$). Finally, since $\sigma>1>2/9$ it follows, by (6.7.13), the results (6.7.51) and (6.7.61) of Lemma 6.7.10 and Lemma 6.7.11, and the definitions (6.7.63) and (6.7.64), that for all $(\nu,p)\in E$ and all $n\in\mathbb{N}$ one has

$$|f_n(\nu,p)|\leq\exp(8\sigma^4)C_f(\varrho,\vartheta)(1+|\mathrm{Im}(\nu)|)^{-\varrho}(1+|p|)^{-\vartheta}\leq 2^{\varrho+5}\exp(8\sigma^4)C_f(\varrho,\vartheta)d^\flat(\nu,p)=d(\nu,p)\ .$$

Since it has been found that all of the hypotheses of Lemma 6.7.9 are satisfied, it therefore follows by that lemma that the sum formula for $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(f)$ is valid ■

The proof of Theorem B. We shall deduce Theorem B from the Weak Sum Formula (Proposition 6.7.2). In order to achieve this we employ the same method as that employed, in similar contexts, on both Page 64 of [5] and Pages 107-109 of [32].

We shall make use of the terminology introduced in the first few paragraphs of the current subsection (up to, and including, the paragraph containing Equation (6.7.4)); our hypotheses concerning ω_1, ω_2, $\mathfrak{a}$, $\mathfrak{b}$, $g_{\mathfrak{a}}$ and $g_{\mathfrak{b}}$ are as stated in the second paragraph of this subsection. We suppose moreover that the set E is as given by the equation (6.7.13) of Lemma 6.7.6, and that σ, ϱ, ϑ and h satisfy the relevant hypotheses of Theorem B, so that $\sigma\in(1/2,1)$, $\varrho,\vartheta\in(3,\infty)$ and $h\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)=\mathcal{H}^\sigma_0(\varrho,\vartheta)$. If it follows that the sum formula for $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(h)$ is valid then Theorem B is true. By means of an application of Lemma 6.7.9 we shall succeed in deducing the validity of the sum formula for $Y^{\mathfrak{a},\mathfrak{b}}_{\omega_1,\omega_2}(h)$, and shall thereby prove Theorem B. For the greater part of this proof we shall be concerned with the preliminary steps that enable this application of Lemma 6.7.9.

Let g be the mapping from $\{\nu\in\mathbb{C} : |\mathrm{Re}(\nu)|\leq\sigma\}\times\mathbb{Z}$ into $\mathbb{C}$ given by

$$g(\nu,p)=\frac{h(\nu,p)}{\left(1-\nu^2\right)^2\left(4-\nu^2\right)^{-2}}\ . \tag{6.7.65}$$

The function $j(\nu)=(1-\nu^2)^2(4-\nu^2)^{-2}$ is meromorphic on $\mathbb{C}$ and even; given that $1/2<\sigma<1$, one has

$$\frac{1}{\sqrt{|j(\nu)|}}=\left|\frac{\nu-2}{\nu-1}\right|\left|\frac{\nu+2}{\nu+1}\right|\leq\left(1+\frac{1}{|\nu-1|}\right)\left(1+\frac{1}{|\nu+1|}\right)\leq 2\left(1+\frac{1}{1-\sigma}\right)<\infty \qquad (|\mathrm{Re}(\nu)|\leq\sigma).$$

Therefore, and since $h\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)$, it follows from the definition (6.7.65) that one has

$$g\in\mathcal{H}^\sigma_\star(\varrho,\vartheta)\ . \tag{6.7.66}$$

For each $n \in \mathbb{N}$, we define $g_n^\flat$ to be the mapping from $\mathbb{C} \times \mathbb{Z}$ into $\mathbb{C}$ given by

$$g_n^\flat(\nu, p) = -i\sqrt{\frac{n}{\pi}} \int\limits_{(0)} g(\xi, p) \exp\left(n(\nu - \xi)^2\right) \mathrm{d}\xi \, . \tag{6.7.67}$$

As is shown by (11.28)-(11.30) of [32], it follows from (6.7.66) and (6.7.67) that, for $n \in \mathbb{N}$, $p \in \mathbb{Z}$ and $\nu \in \mathbb{C}$, one has

$$\begin{aligned}\left|g_n^\flat(\nu, p)\right| \leq \sqrt{\frac{n}{\pi}} \int\limits_{-\infty}^{\infty} \left|g(i\tau, p) \exp\left(n(\nu - i\tau)^2\right)\right| \mathrm{d}\tau \ll_{g,\varrho,\vartheta} \\ \ll_{g,\varrho,\vartheta} \ (1 + |\mathrm{Im}(\nu)|)^{-\varrho} (1 + |p|)^{-\vartheta} \exp\Big(n \left(\mathrm{Re}(\nu)\right)^2\Big) \, .\end{aligned} \tag{6.7.68}$$

In particular, the integral on the right-hand side of Equation (6.7.67) is absolutely convergent when $n \in \mathbb{N}$, $p \in \mathbb{Z}$ and $\nu \in \mathbb{C}$. Indeed, using only the fact that (by virtue of (6.7.66) and the hypothesis that $\varrho, \vartheta \in (3, \infty)$) the function $(\tau, p) \mapsto |g(i\tau, p)|$ is bounded on $\mathbb{R} \times \mathbb{Z}$, one can show that when $n \in \mathbb{N}$, $p \in \mathbb{Z}$ and $T \in (0, \infty)$ the integral on the right-hand side of Equation (6.7.67) converges uniformly for all complex ν such that $\max\{|\mathrm{Re}(\nu)|, |\mathrm{Im}(\nu)|\} \leq T$. It therefore follows, by (for example) Section 2.83 and Section 2.84 of [43], that

$$\text{for } n \in \mathbb{N} \text{ and } p \in \mathbb{Z} \text{ the mapping } \ \nu \mapsto g_n^\flat(\nu, p) \ \text{ is holomorphic on } \mathbb{C}. \tag{6.7.69}$$

By means of the change of variable $\xi = -\phi$ (say), and by the application of the identity $g(-\phi, p) = g(\phi, -p)$ (inferred from (6.7.66)), one finds that it moreover follows from (6.7.67) that one has

$$g_n^\flat(\nu, p) = g_n^\flat(-\nu, -p) \qquad\qquad (n \in \mathbb{N},\ p \in \mathbb{Z} \text{ and } \nu \in \mathbb{C}). \tag{6.7.70}$$

We shall later have need of two further properties of the sequence $(g_n^\flat)_{n\in\mathbb{N}}$. The properties in question are, firstly, that

$$\lim_{n\to\infty} g_n^\flat(\nu, p) = g(\nu, p) \qquad\qquad (p \in \mathbb{Z},\ \nu \in \mathbb{C} \text{ and } |\mathrm{Re}(\nu)| \leq \sigma), \tag{6.7.71}$$

and, secondly, that

$$\sup\Big\{(1 + |\mathrm{Im}(\nu)|)^{\varrho} (1 + |p|)^{\vartheta} \left|g_n^\flat(\nu, p)\right| \ : \ n \in \mathbb{N},\ p \in \mathbb{Z},\ \nu \in \mathbb{C} \ \text{and} \ |\mathrm{Re}(\nu)| \leq \sigma\Big\} < \infty \, . \tag{6.7.72}$$

The proofs of both of these properties depend on the fact (used in both [5] and [32]) that, for $n \in \mathbb{N}$, $p \in \mathbb{Z}$ and $\nu \in \mathbb{C}$, one has

$$\int\limits_{(0)} g(\xi, p) \exp\left(n(\nu - \xi)^2\right) \mathrm{d}\xi = \int\limits_{(\alpha)} g(\xi, p) \exp\left(n(\nu - \xi)^2\right) \mathrm{d}\xi \qquad\qquad (-\sigma \leq \alpha \leq \sigma). \tag{6.7.73}$$

By (6.7.67) and the case $\alpha = \mathrm{Re}(\nu)$ of (6.7.73), one obtains

$$g_n^\flat(\alpha + it, p) = \sqrt{\frac{n}{\pi}} \int\limits_{-\infty}^{\infty} g(\alpha + i\tau, p) \exp\left(-n(t - \tau)^2\right) \mathrm{d}\tau \qquad (n \in \mathbb{N},\ p \in \mathbb{Z},\ t \in \mathbb{R} \text{ and } -\sigma \leq \alpha \leq \sigma). \tag{6.7.74}$$

Given (6.7.66), the property (6.7.71) follows by (6.7.74) and the equation 1.17.6 of [38], while the property (6.7.72) follows by (6.7.74), (6.7.66) and the upper bound deduced in (11.29) of [32]. We remark that the result (6.7.73) is a corollary of the relation (6.7.66), which implies that the relevant mappings $\xi \mapsto g(\xi, p)$ are holomorphic on a neighbourhood of the strip $\{\xi \in \mathbb{C} : |\mathrm{Re}(\xi)| \leq \sigma\}$, and that in that strip one has

$$g(\xi, p) \ll_{g,\varrho,\vartheta} \ (1 + |\mathrm{Im}(\nu)|)^{-\varrho} (1 + |p|)^{-\vartheta} \leq (1 + |\mathrm{Im}(\nu)|)^{-3} \, .$$

We now define $d^\flat$ and $h_1^\flat, h_2^\flat, \ldots$ to be the mappings from $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq 3/2\} \times \mathbb{Z}$ into $\mathbb{C}$ given by

$$h_n^\flat(\nu,p) = \left(1-\nu^2\right)^2 \left(4-\nu^2\right)^{-2} g_n^\flat(\nu,p) \qquad (n \in \mathbb{N},\ p \in \mathbb{Z},\ \nu \in \mathbb{C} \text{ and } |\mathrm{Re}(\nu)| \leq 3/2), \tag{6.7.75}$$

$$d^\flat(\nu,p) = \left(1-\nu^2\right)^2 \left(4-\nu^2\right)^{-(\varrho+4)/2} (1+|p|)^{-\vartheta} \qquad (p \in \mathbb{Z},\ \nu \in \mathbb{C} \text{ and } |\mathrm{Re}(\nu)| \leq 3/2). \tag{6.7.76}$$

We recall that the function $j(\nu) = (1-\nu^2)^2(4-\nu^2)^{-2}$ is meromorphic on $\mathbb{C}$ and even. One has, moreover,

$$\sqrt{|j(\nu)|} = \frac{|\nu-1|}{|\nu-2|}\,\frac{|\nu-(-1)|}{|\nu-(-2)|} \leq (1)(1) = 1 \qquad (|\mathrm{Re}(\nu)| \leq 3/2) \tag{6.7.77}$$

and $j'(1) = j(1) = 0$. Consequently it follows from (6.7.68), (6.7.69), (6.7.70) and (6.7.75) that one has

$$h_n^\flat \in \mathcal{H}_\star^{3/2}(\varrho,\vartheta) \qquad (n \in \mathbb{N}). \tag{6.7.78}$$

Therefore Proposition 6.7.2 implies that

$$\text{for } n \in \mathbb{N} \text{ the sum formula for } Y_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}\left(h_n^\flat\right) \text{ is valid.} \tag{6.7.79}$$

At the same time, it follows by (6.7.76) and the case $\sigma = 3/2$ of Lemma 6.7.11 that

$$d^\flat \in \mathcal{H}_\star^{3/2}(\varrho,\vartheta)\ , \tag{6.7.80}$$

$$d^\flat(\nu,p) \geq 2^{-(\varrho+5)} (1+|\mathrm{Im}(\nu)|)^{-\varrho} (1+|p|)^{-\vartheta} \qquad (\,(\nu,p) \in E) \tag{6.7.81}$$

and

$$\text{the sum formulae for } Y_{\omega_1,\omega_1}^{\mathfrak{a},\mathfrak{a}}\left(d^\flat\right) \text{ and } Y_{\omega_2,\omega_2}^{\mathfrak{b},\mathfrak{b}}\left(d^\flat\right) \text{ are valid.} \tag{6.7.82}$$

For $n \in \mathbb{N}$, we define h_n to be the restriction of the mapping $h_n^\flat$ to the set $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{Z}$. Similarly, we define $d^\sharp$ to be the restriction of the mapping $d^\flat$ to the set $\{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{Z}$. Since $3/2 > 1 > \sigma > 1/2 > 2/9$, it follows by (6.7.78), (6.7.79), (6.7.80), (6.7.81), (6.7.82), the definition (6.7.13) and what is observed in our Remark 1.9.2 that

$$d^\sharp \in \mathcal{H}_\star^{\sigma}(\varrho,\vartheta) = \mathcal{H}_0^{\sigma}(\varrho,\vartheta)\ , \tag{6.7.83}$$

$$d^\sharp(\nu,p) \geq 2^{-(\varrho+5)} (1+|\mathrm{Im}(\nu)|)^{-\varrho} (1+|p|)^{-\vartheta} \qquad (\,(\nu,p) \in E), \tag{6.7.84}$$

$$\text{the sum formulae for } Y_{\omega_1,\omega_1}^{\mathfrak{a},\mathfrak{a}}\left(d^\sharp\right) \text{ and } Y_{\omega_2,\omega_2}^{\mathfrak{b},\mathfrak{b}}\left(d^\sharp\right) \text{ are valid} \tag{6.7.85}$$

and

$$h_n \in \mathcal{H}_\star^{\sigma}(\varrho,\vartheta) \qquad (n \in \mathbb{N}), \tag{6.7.86}$$

and that

$$\text{for } n \in \mathbb{N} \text{ the sum formula for } Y_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}\left(h_n\right) \text{ is valid.} \tag{6.7.87}$$

Given our definition of the mappings $h_1, h_2, \ldots$, and given the definitions in (6.7.65) and (6.7.75), it moreover follows by (6.7.71), (6.7.72) and (6.7.77) that one has both

$$\lim_{n\to\infty} h_n(\nu,p) = h(\nu,p) \qquad (p \in \mathbb{Z},\ \nu \in \mathbb{C} \text{ and } |\mathrm{Re}(\nu)| \leq \sigma) \tag{6.7.88}$$

and

$$\sup\left\{(1+|\mathrm{Im}(\nu)|)^{\varrho} (1+|p|)^{\vartheta} \left|h_n(\nu,p)\right| \ :\ n \in \mathbb{N}, p \in \mathbb{Z}, \nu \in \mathbb{C} \text{ and } |\mathrm{Re}(\nu)| \leq \sigma\right\} < \infty\ . \tag{6.7.89}$$

Note that, by (6.7.89), the condition (6.7.6) of Lemma 6.7.3 is satisfied when one has $f_n = h_n$ for all $n \in \mathbb{N}$.

Just one further definition will put us in a position to apply Lemma 6.7.9. We define the mapping $d : \{\nu \in \mathbb{C} : |\mathrm{Re}(\nu)| \leq \sigma\} \times \mathbb{C} \to \mathbb{C}$ by setting

$$d = 2^{\varrho+5} C_h^*(\varrho, \vartheta)\, d^\sharp \;, \tag{6.7.90}$$

where $C_h^*(\varrho, \vartheta)$ denotes the non-negative real number equal to the supremum on the left-hand side of the inequality (6.7.89). It then follows, by linearity, from (6.7.83) and (6.7.85) that

$$d \in \mathcal{H}_\star^\sigma(\varrho, \vartheta) \text{ and the sum formulae for } Y_{\omega_1,\omega_1}^{\mathfrak{a},\mathfrak{a}}(d) \text{ and } Y_{\omega_2,\omega_2}^{\mathfrak{b},\mathfrak{b}}(d) \text{ are valid.} \tag{6.7.91}$$

By (6.7.89), the definition of $C_h^*(\varrho, \vartheta)$, (6.7.84) and (6.7.90) one has, moreover,

$$\left|h_n(\nu, p)\right| \leq C_h^*(\varrho, \vartheta)\,(1 + |\mathrm{Im}(\nu)|)^{-\varrho}\,(1 + |p|)^{-\vartheta} \leq d(\nu, p) \qquad (n \in \mathbb{N} \text{ and } (\nu, p) \in E). \tag{6.7.92}$$

By our initial hypotheses in this proof, and by (6.7.86), (6.7.87), (6.7.88), (6.7.89), (6.7.91) and (6.7.92), it follows that the hypotheses of Lemma 6.7.9 are satisfied if one substitutes h and the sequence $(h_n)_{n\in\mathbb{N}}$ for the function f and sequence $(f_n)_{n\in\mathbb{N}}$ of that lemma (while taking d to be given by (6.7.90)). Lemma 6.7.9 therefore implies that the sum formula for $Y_{\omega_1,\omega_2}^{\mathfrak{a},\mathfrak{b}}(h)$ is valid. This conclusion means that the results of Theorem B are obtained; since it is a conclusion that has been arrived at independently of any assumptions other than the stated hypotheses of Theorem B, our proof of that theorem is now complete ■

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